PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The Yield Point and Initial Stages of Plastic Strain in Mild Steel Subjected to Uniform and Non-Uniform Stress Distributions

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ERRATA.

Page 112. Title of Fig. 4. For *Torsion* read *Tension.*

Page 113. Title of Fig. 5. For *Torsion* read *Tension.*

Page 114. Title of Fig. 6. For *Torsion* read *Tension.*

Page 123. Title of Fig. 11 should read *Apparatus for Straining in Flexure.*

Page 123. Title of Fig. 12 should read *Bending Moment—Curvature Diagrams for Flexure.*

Page 124. Line 5. For *Fig.* 12 read *Fig.* 11.

Page 130. Line 4. For *Fig.* 14 read *Fig.* 19.

Page 133. Line 9. For *Fig.* 17 read *Fig.* 16.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

IV. *The Yield Point and Initial Stages of Plastic Strain in Mild Steel Subjected to Uniform and Non-Uniform Stress Distributions.*

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Summary.

The paper describes an investigation carried out to determine for three samples of mild steel, (1) the relation between the stress at the yield point in simple tension and in the non-uniform distributions produced by torsion, flexure and internal pressure in a hollow cylinder, and (2) the stress distribution in each of the latter cases in the early stages of overstrain.

Apparatus is described for obtaining load-deformation diagrams in which the true resistance to deformation during overstrain is measured. Those obtained for the non-uniform distributions are compared with theoretical diagrams based upon the assumption that a specific shear stress causes the initial breakdown, and that the initial stages of plastic strain take place at a uniform, but lower, shear stress.

It is shown that the maximum shear stress at the initial yield point is consistently higher in the non-uniform distributions than in uniform tension. In the cylinders a pronounced scale effect was observed. All the results are consistent with the supposition that the initial dislocation resulting in elastic breakdown takes place at a critical value of the shear stress at a certain depth in the material; in other words, that a surface layer exists possessing the same elastic properties as, but a higher elastic limit than, the interior.

In the initial stages of overstrain, the load-deformation diagrams follow closely the theoretical shape, and confirm the assumption that a lower stress exists in the over-strained parts than is required to initiate the yielding process.

The author suggests that the stress reduction is a consequence of variations in load from one crystal grain to another. Irreversible and discontinuous displacements occurring either at the crystal boundary or in the interior of the crystal, which are

associated with the process of slip, are regarded as affording relief to the reversible elastic distortion in the same crystal, the amount of relief depending upon the magnitude of the individual displacements.

1. *Introduction.*

The stresses in structural members or machine parts carrying loads are rarely uniformly distributed in the material. The non-uniformity here considered is not that arising from lack of homogeneity due to a crystal aggregate, but that due to the character of the applied loads. The former, although giving possible stress differences as between neighbouring crystal grains, will disappear statistically when any considerable area is considered. In the latter, however, the variation is continuous, and the stress usually reaches a maximum value at a boundary surface. The commonest cases of this type of stress variation are those produced by torsion, flexure, internal pressure in a thick hollow cylinder, contact pressure, and those due to discontinuities in shape.

The purpose of the present paper is to describe an investigation into the conditions under which elastic failure takes place in certain of these cases, and also to determine, as far as is possible, the changes which take place in the stress distribution when the elastic limit is exceeded. Three of the cases enumerated above, namely, torsion, flexure and internal pressure are particularly suitable for this study since, assuming a perfectly elastic material, the stress distribution produced by given loads can be calculated accurately. They are, moreover, the only cases where both the applied loads and the resulting displacements may be conveniently measured experimentally.

The conditions governing elastic breakdown in ductile materials have been the subject of many elaborate investigations. Many theories have been based upon the results of these investigations, and experimental results can be adduced in support of all of them.* Unless the refinement of the experimental methods, or the legitimacy of calculations based upon the results, are called into question, the diversity of the theories proposed is evidence that some factor has not been taken into account; some factor, that is, not directly related to the magnitude of stress and strain. The conditions are, in fact, very imperfectly understood.

The theories of failure in the case of ductile materials which have been most widely accepted in this country are the shear stress theory, usually known as Guest's law, and the strain energy theory proposed by Haigh.† According to the former the elastic limit is reached at a certain definite value of the maximum shear stress in the material. That is to say, it is dependent upon the difference of the greatest and least principal stresses, and not upon their absolute magnitudes, or upon any elastic constant of the material. The strain energy theory, on the other hand, regards the value of

* A comprehensive review of these theories and experimental investigations has been given by W. Lode, V.D.I. Forschungsheft 302 (1927).

† 'British Association Report,' 1919, p. 486.

the strain energy function as the criterion, and this is dependent upon the absolute value of each of the three principal stresses, and requires also the elastic constants for its specification.

If a specific shear stress is the criterion of failure, it would be expected that the elastic limit in torsion would be reached at a value of the maximum shear stress equal to one-half the tensile stress at the elastic limit in simple tension. Tension and torsion tests are matters of common practice, and a large amount of data is available for making the comparison. Although a study of the records of tests carried out on the same material reveals a wide variation in the ratio of the stresses at the elastic limit, the evidence is almost unanimous that the shear stress in torsion is greater than the shear stress in tension. This may be illustrated by reference to a few of the more recently published tests. SEELY and PUTNAM,* in an investigation carried out with the sole object of studying the relation between tension and torsion tests, give results which show values of the ratio of shear stress in torsion to shear stress in tension ranging from 1·31 to 1·59 for solid specimens, and 1·05 for hollow specimens, the material used being mild carbon and alloy steels. SEIGLE and CRETIN† give values of about 1·3 for mild steel, while results obtained by HANKINS, HANSON and FORD‡ on heat-treated spring steels range from 1·27 to 1·77, lying chiefly between 1·4 and 1·6. A similar wide variation is shown in tests by HATFIELD§ on various steels, values ranging from 1·33 to 1·95 being obtained.

In simple uniform flexure the stress distribution (assuming perfect elasticity) is linear as in torsion, but every part of the material is in simple tension or compression. It would be expected therefore that, whatever theory of elastic failure be adopted, such failure should take place in flexure when the maximum tensile or compressive stress in the beam reached the value observed at failure in a simple tension or compression test. The results of flexure tests, however, show stresses substantially greater than those in tension. This fact has been well known for many years in the case of cast iron,|| although in this material it may be ascribed, partly at all events, to imperfect elasticity and to a difference in properties as between tension and compression; but, as UNWIN has pointed out,¶ these are insufficient to account completely for the difference. KENNEDY** appears to have been the first to call attention to the discrepancy in the case of mild steel. He carried out a number of tests on rectangular beams, applying both a uniform bending moment and also a central load, and found that the ratio of the maximum fibre stress in flexure to that in tension at what was taken as the yield

* 'University of Illinois Bulletin No. 115,' November, 1919.

† 'Revue Met,' June, 1925.

‡ 'J. Iron and Steel Inst.,' 1926, II, p. 265.

§ 'Proc. Inst. Mech. E.,' May, 1919.

|| See, for example, BARLOW, 'Phil. Trans.,' 1855, p. 225, and 1857, p. 463.

¶ 'The Engineer,' vol. 62, p. 351 (1886).

** 'Engineering,' vol. 115 (1923).

point was 1·46 to 1·64 for uniform bending moment, and 1·55 to 1·98 for central loading. The selection of the point of failure in these tests is open to some criticism, and the effect may be somewhat exaggerated. Tests carried out by SCOBLE* on round bars of steel under uniform bending moment gave values ranging from 1·14 to 1·40.

In the case of thick hollow cylinders under internal hydrostatic pressure, there is a state of compound stress at the internal surface, the radial and tangential components, and also the shear stress having their maximum values there. In tests carried out on mild steel cylinders† the shear stress at the yield point has been found to be as much as 20 per cent. in excess of that deduced from the tension test.

There is therefore definite evidence in each of the cases of non-uniform stress cited above that the shear stress at failure is greater than that found in the tension test.

It will be convenient at this point to examine in some detail two arguments occasionally used to account for the discrepancy. In the first place, for many materials the elastic limit cannot be located within very close limits. For most steels other than normalised or annealed mild steel, a gradual change of slope takes place in the load-strain diagram at the elastic limit in tension, so that the exact location is somewhat indefinite. In a torsion, flexure, or thick cylinder test the first deviation from a linear relation between load and displacement will occur simultaneously with the first deviation from a linear relation between stress and strain at the surface, and owing to the character of the distribution of the stress over the cross section, it can readily be shown that if the elastic limit is not very well defined in simple stress and strain it will be much less well defined in such cases of non-uniform stress. If, for instance, the change of slope of the stress-strain diagram at the elastic limit in tensile stress and strain is 10 per cent., the corresponding change in torsion will be as small as 2 per cent., and might indeed escape observation unless the most refined methods were employed. That this may be the explanation of some of the high values obtained, particularly for alloy steels, is not improbable. In a material such as normalised or annealed mild steel, however, where the elastic limit usually coincides with the yield point, this explanation does not hold. The breakdown occurs very suddenly, and there is no uncertainty as to its position on the stress-strain diagram.

The second argument which may be examined here is whether any error is involved in the usual methods of calculation of the stresses in torsion and flexure. The assumptions underlying the ordinary methods of calculation are (1) that plane sections in the unstrained bar remain plane; (2) that HOOKE's law is obeyed, and that the elastic constants are the same in tension and compression.

In the case of simple torsion of a straight bar of uniform circular cross section, and in the flexure of a bar by uniform bending moment, it has been shown‡ mathematically that plane sections do remain plane, and consequently that the distribution of strain

* 'Engineering,' vol. 123 (1927).

† COOK and ROBERTSON, 'Engineering,' vol. 92, p. 786 (1911).

‡ LOVE, 'Theory of Elasticity,' 3rd ed., p. 127.

is linear. If the material is also perfectly elastic, the stress distribution must be linear, and the ordinary calculation will be accurate. The elasticity of no metal is, however, perfect in the mathematical sense, and it will be instructive to estimate the effect of a deviation from a linear relation between stress and strain. In many materials this deviation is extremely small, but in others it is appreciable before the yield point is reached. In such cases the strain increases more rapidly than the stress, and the stress-strain curve in simple tension or shear may be more closely represented by a parabola OA (fig. 1) having the equation

$$p = (p_1 + c)\, e/e_1 - c\,(e/e_1)^2,$$

where p_1, e_1 are the stress and strain corresponding to some point A, and c is the intercept on the stress axis of the tangent to the curve at A. The value of c is easily obtained in any given case knowing the shape of the curve OA. When multiplied by e_1 it is equal to one-half of the change in modulus between the points O and A.

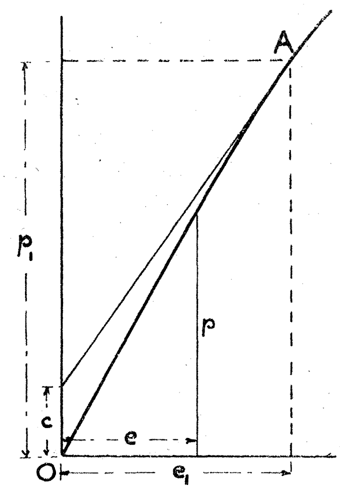

FIG. 1.

If a cylindrical bar be twisted so that the maximum shear strain is e_1, it can be shown by simple integration that the torque required is

$$T = \tfrac{1}{2}\pi r^3\,(p_1 + \tfrac{1}{5}c),$$

and if therefore a measurement of the torque is made, the maximum stress deduced from it by the ordinary calculation will be in excess of the true stress by $c/5$. Similarly it may be shown that in the flexure of a rectangular beam by uniform bending moment

$$M = \tfrac{1}{6}bd^3\,(p_1 + \tfrac{1}{4}c),$$

b and d being the breadth and depth, and in that of a beam of circular cross section

$$M = \tfrac{1}{4}\pi r^3\,\{p_1 + (1 - 32/15\pi)\,c\},$$

that is to say the error in the estimation of stress from bending moment is $c/4$ and very approximately $c/3$ for these cross sections. The quantity c is generally small, at all events for ductile ferrous metals. It is extremely small for annealed or normalised mild steel.* It follows that the error made in deducing the stress from the torque or bending moment is negligible, and will not account for the difference between the results in tension, and those in torsion and flexure. The same can legitimately be said of the hollow cylinder under pressure.

One must, therefore, regard the shear stress theory, in its simple form, as inadequate. It does not stand the test of an examination of the simplest cases of shear stress in

* For the mild steels used in the present investigation the value of c corresponding to the yield point was not greater than 0·1 ton per square inch, and was barely measurable.

practice, namely, those produced by simple tension and simple torsion. This does not imply that shear stress is not the cause of failure. The results of the experiments to be described in this paper confirm, in the author's opinion, the prevailing view that failure is indeed by shear stress, but it is not necessarily the shear stress existing *at the surface*.

It is interesting to note, however, that the strain energy theory of HAIGH gives results which are in satisfactory agreement with torsion tests. The strain energy per unit volume at the surface of a twisted bar is equal to $s^2/2C$ where s is the shear stress at the surface, and C is the modulus of rigidity. If f is the stress at the elastic limit in tension, according to this theory failure should occur in torsion when

$$s^2/2C = f^2/2E,$$
$$s = f\sqrt{C/E}.$$

For steels, E and C have values about 30,000,000 lbs. square inch and 11,700,000 lbs. square inch respectively, so that

$$s = 0 \cdot 625 f = 1 \cdot 25 \left(\tfrac{1}{2} f\right).$$

Thus the shear stress in torsion should be 1·25 times that in tension at the elastic limit. This figure is of the order frequently found by experiment. Professor HAIGH* has also shown that the results of the thick cylinder tests by the author and Professor ROBERTSON are consistent with the strain-energy theory of failure. But it gives no help in the case of flexure, where indeed the shear stress theory as ordinarily applied, and all other theories so far proposed, break down.

To account for the anomaly in flexure, SCOBLE suggested that the more heavily stressed fibres were supported, or reinforced, by those more lightly stressed and by this means enabled to sustain a greater stress before failure than under the uniform stress found in the tension test. It is difficult to imagine by what process this support can be given. It requires an interaction—which must be of the nature of stress—between adjacent layers, which is impossible according to the mathematical theory of simple flexure. Yet it is specially interesting as suggesting that another factor, not hitherto recognised, may affect the conditions producing elastic breakdown; not only the magnitude of stress and strain, but also their distribution.

It should be mentioned that while a considerable amount of information is available regarding the relative values of the yield stress in tension and torsion, tension and flexure and tension and internal pressure in thick cylinders, there appear to be no published results of tests carried out upon the same material under all these stress systems.

When the yield point is passed it is obvious that the stress distribution must be completely altered, but little attention appears to have been devoted to the nature of this

* *Loc. cit.* p. 493.

alteration. In some discussions of its character* the assumption has been made that in the overstrained regions the stress remains uniform and equal to the initial yield stress. It has been shown,† however, that in the case of mild steel this assumption is not consistent with the shapes of the load-strain diagrams for torsion and flexure. The characteristic shape of the stress-strain diagram in tension for this material has, for torsion, flexure, and thick cylinders, consequences which do not appear yet to have been recognised.

The purposes of this investigation are therefore (1) to examine experimentally the relation between the tensile yield point and the yield point in torsion, flexure, and by internal pressure for the same mild steel, and (2) to determine as far as possible the character of the stress distribution in each of these cases during the process of overstrain.

The first of these presents little difficulty beyond the provision of suitable straining appliances, and strain measuring instruments. The second will necessarily be an indirect process. The experimental evidence will be a load-deformation diagram. Theoretical diagrams will be obtained for an assumed stress distribution based upon the stress-strain diagram obtained in the simple tension test. A comparison will then be made between the theoretical and experimental diagrams.

2. *Materials Used.*

Mild steel is a material particularly suitable for a study of the conditions under which failure takes place under non-uniform stress for two reasons. In the first place, in the normalised or annealed condition it is almost perfectly elastic for stresses below the yield point and, as has been indicated above, in this region there can be little, if any, error in the stresses calculated by the usual methods. Secondly, the yield point is well marked, and generally coincides with the elastic limit.

Three samples of mild steel were used for the tests, the compositions and mechanical properties (as received) being shown in Table I. The material was supplied in the form of rolled bar $1\frac{3}{8}$ in. diameter.

Steels A and B are the standard dead mild steel used in various researches by the Aeronautical Research Committee. They were stated to be of the same composition, but steel B was supplied at a much later date than steel A, and was found to possess different properties. Through the courtesy of Dr. W. H. Hatfield, steel C was specially prepared and supplied by Messrs. Thos. Firth and Sons, Sheffield.

Heat Treatment.—In the tests of specimens subjected to non-uniform stress it is essential that the material at the surface should be free from any overstrain due to machining processes. It was accordingly decided to carry out the heat treatment—normalising or annealing—*after* machining, and in order to avoid oxidation of the

* Muir and Binnie, 'Engineering,' vol. 122, p. 743 (1926); Turner, 'Trans. Camb. Phil. Soc.,' vol. 21, p. 377 (1910).

† A. Robertson and G. Cook, 'Proc. Roy. Soc.,' A, vol. 88 (1913), p. 462.

TABLE I.—Composition and Mechanical Properties (as received) of Mild Steels used in the Tests.

Steel.	C.	Si.	Mn.	S.	P.	Ni.	Yield Stress (tons sq. in.).	Ult. Strength (tons sq. in.).	Elong. Percentage on Length. $4\sqrt{A}$.	Contraction of Area.
	per cent.	per cent.	per cent.	per cent.	per cent.	per cent.				per cent.
A	0·13	0·18	0·70	0·042	0·046	—	20·0	30·6	40·0	72
B	0·13	0·18	0·70	0·042	0·046	—	20·2	28·3	41·4	69
C	0·21	0·09	0·65	0·018	0·018	0·11	18·0	31·6	37·6	62

surface, damage to screw threads, etc., the heat treatment was carried out *in vacuo*. The specimens were put into a silica tube, which was exhausted by means of a Cenco Hyvac pump capable of producing a vacuum of less than 0·001 mm. The tube was placed in an electric furnace, and the temperature observed by a platinum platinum-rhodium thermocouple placed as close as possible to the vacuum tube in the neighbourhood of the specimen. Steels A and B were heated slowly to 900° C., and maintained at that temperature (as recorded by the thermocouple) for half an hour. This is rather longer than might appear necessary for a normalising process, but owing to the fact that the surface remained untarnished, and was consequently highly reflecting, the specimen would take longer to reach the furnace temperature. The tube was then withdrawn, and the specimen allowed to cool by radiation.

In the case of steel C the specimens were allowed to cool in the furnace. Some preliminary tests seemed to indicate that the yield point of this steel was not entirely unaffected by the rate of cooling, and as the rate of cooling in the normalising process was greater in the small tension and torsion specimens than in the cylinders, it was considered better to avoid lack of uniformity from this cause by allowing the cooling to take place in the furnace, which, owing to the mass of material involved, took place at a rate little affected by the size of the specimen.

3. *Tension.*

It is well known that in mild steel the yield point in tension is marked by a sudden reduction in the load required to continue the straining process. Experiments* have shown that in tensile tests the reduction in load may amount in some cases to over 30 per cent. A typical load strain diagram is shown in fig. 2. The load corresponding to the point A is the yield point observed in an ordinary tensile test. The beam of the testing machine falls, and remains on the stop while the strain increases until the point B is reached, when further strain raises the beam again, and an increase of strain requires

* ROBERTSON and COOK, *loc. cit.*

an increase of load. Between the points A and B a smaller load is required to continue the strain than was required to produce the original yield. The true shape of this portion of the diagram is only obtained when the actual load on the specimen is automatically recorded, either by a weighbar in series or other device. It is found that for a total strain quite large compared with the original elastic strain, the load remains sensibly constant at a reduced value, which may be described as the "lower" yield load to distinguish it from the initial or "upper" yield load at which breakdown first took place.

For the determination of the initial and lower yield points in simple tension, test pieces having their axes both in the longitudinal direction and in the transverse direction were prepared from each of the steels. The shape and dimensions of these specimens are shown in fig. 3. As the diameter of the rolled bar was only $1\frac{3}{8}$ in., the transverse

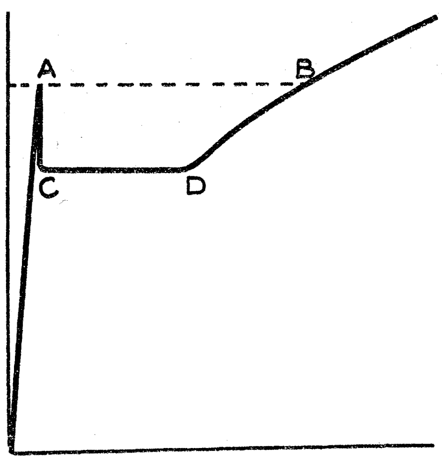

Fig. 2.

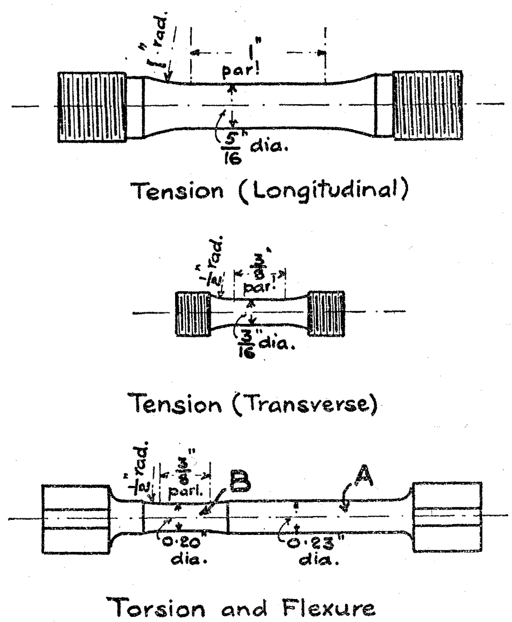

Fig. 3.—Test Pieces for Tension, Torsion and Flexure.

specimens were necessarily very short. The longitudinal specimens were taken from the material between the centre and the surface of the bar and the central parallel portion of all specimens was connected to the screwed ends by a fillet of large radius.

In order to measure the lower yield stress accurately throughout the complete range of strain it is essential that both the application and measurement of the load applied to the specimen should be carried out by such means that at all instants the stress in the specimen is balanced. It is impossible to secure this condition completely, because the breakdown at the initial yield point takes place suddenly, and in any type of loading appliance there is insufficient rigidity to prevent entirely the sudden extension of the specimen under elastic forces left momentarily unbalanced in the straining mechanism.

The weighbar method practically eliminates the difficulty as far as the measurement of the load is concerned. This method was probably first employed by KENNEDY* and has since been used by DALBY, ROBERTSON, and others. It consists simply of measuring by means of an extensometer, the elastic extension of a steel bar placed in series with the test specimen. The device forms, in fact, a spring balance of the greatest stiffness that it is possible to obtain. It can be made as sensitive as desired by refinements in the extensometer, and it can be calibrated readily by dead load in a testing machine. The arrangement used by the author is shown diagrammatically in fig. 4. The weighbar B and the test-piece A are held accurately in line by a connecting nut. The extension of the weighbar and consequently the load on the specimen is measured by the angular movement of the mirror C and any error due to displacement of the system as a whole is avoided by using a second mirror C′ attached rigidly to the lower frame of the extensometer, and observing the scale by double reflection. The extensometer was not removed from the weighbar after calibration, which was made with a scale distance the same as that used in the tests.

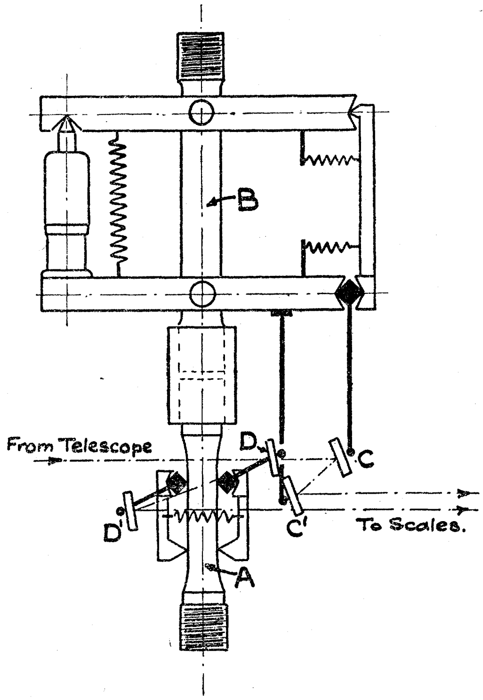

FIG. 4.—Arrangement for Load-Strain Measurement in Torsion.

The extension of the test-piece itself was measured by a simple form of mirror extensometer, the principle of which will be readily seen from fig. 4. Again, a double reflection from mirrors D and D′ was used, and the position was so arranged that a single telescope was sufficient to observe both the weighbar scale and the test-piece scale at the same time, the scales appearing side by side in the field of view. The absolute magnitude of the strain in the specimen was not desired. In the elastic range it could be inferred from the stress, YOUNG's Modulus being known from other tests. What it was desired to measure was the ratio of the strain after the yield point had been passed to that at the yield point.

One division on the weighbar scale corresponded to a load of 0·0125 ton, and it was possible to read to one-tenth of a division. The stress in the specimen ($\frac{5}{16}$-inch diameter) could therefore be measured accurately to within 0·02 ton per square inch.

The apparatus used for applying the strain is shown in fig. 5. The weighbar and test-piece were held between axial loading grips of the ball type designed by Professor ROBERTSON and the author. The importance of ensuring truly axial loading in accurate determinations of the primitive yield point in tension and compression has frequently

* 'Proc. Inst. Mech. Eng.' (1886), p. 63.

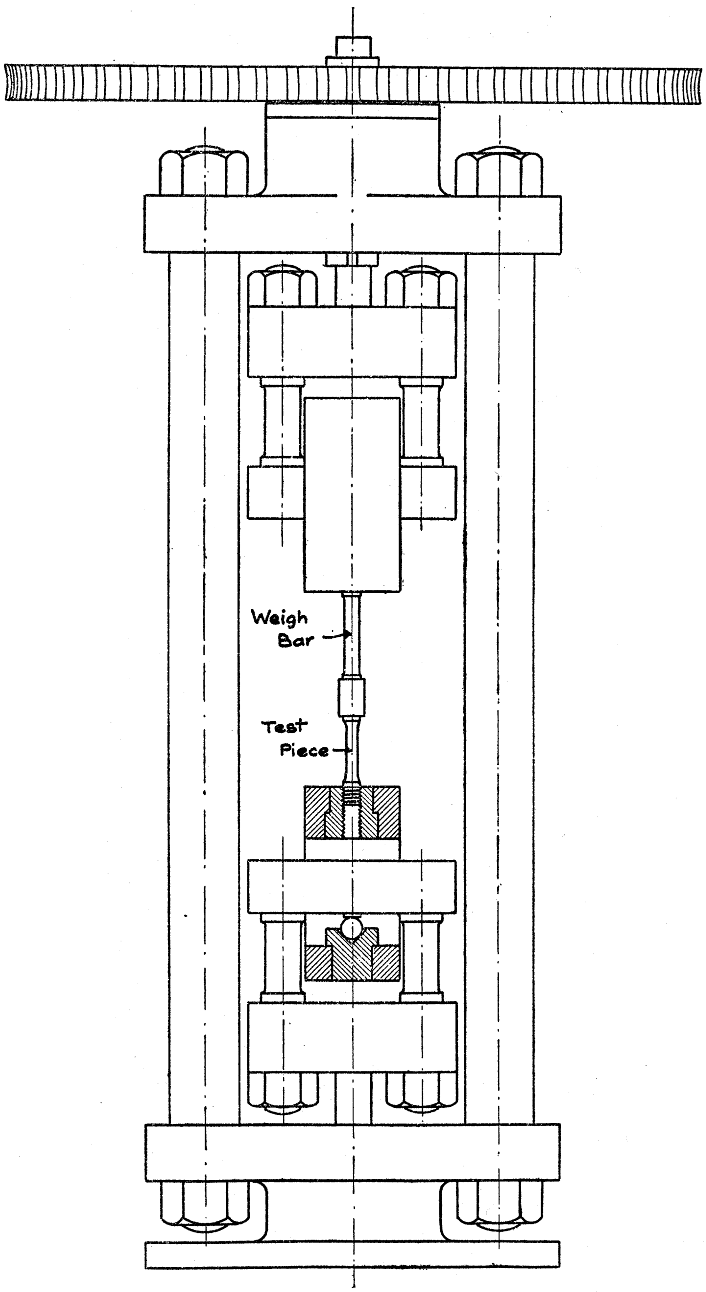

FIG. 5.—Apparatus for Straining in Torsion.

been emphasised; some sacrifice of rigidity is, however, involved by the use of these grips. The strain was applied by means of a screw and worm gear, which was driven by a motor through speed reduction gear such that the time required to reach the yield point was about 30 minutes.

In each of the materials used the elastic limit coincided with the yield point, there being no evidence of any appreciable deviation from a linear stress-strain relation until that point was reached. The yield occurred very suddenly in each case, but particularly so in steel B. The action in this material resembled the sudden release of a spring, and the drop from the initial to the lower yield stress appeared to be almost instantaneous. It is interesting to note that this steel had a much higher yield point after normalising *in vacuo* than in the "as received" condition.

Typical diagrams for the longitudinal tests are shown in fig. 6. Strain measurements were not made in the transverse tests on account of the small lengths available. The numerical results are given for all the tests in Table II. Owing to the small but unavoidable elasticity of the straining mechanism, it was not possible to determine the true

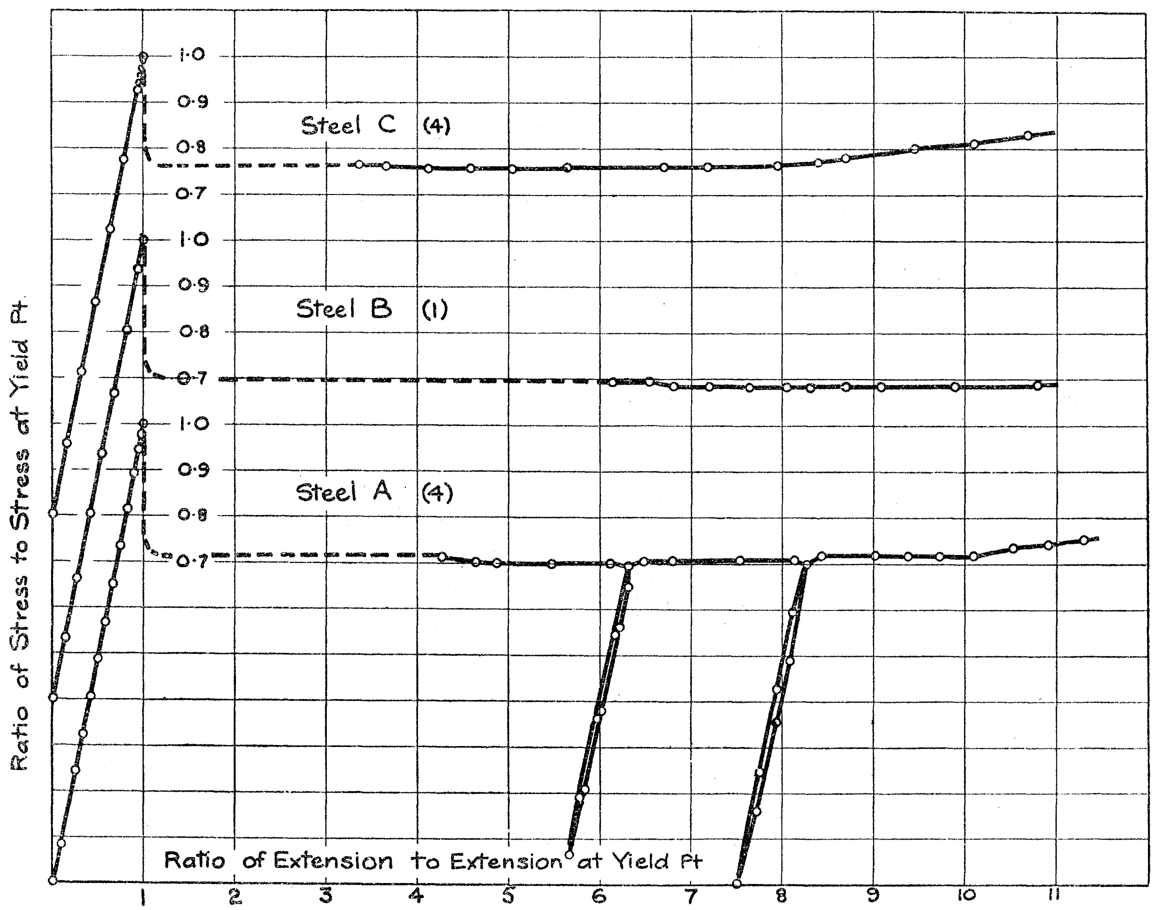

FIG. 6.—Load Extension Diagrams for Torsion (Longitudinal).

shape of the stress-strain curve immediately after the yield point. The manner in which the yield took place suggested very strongly that the transition from the upper to the lower yield stress was almost instantaneous, and that the relation would be more correctly represented in the manner shown by the broken line in the diagram in fig. 6 than by a gradually sloping line. The lower yield stress is seen to remain constant over a range of strain several times as great as the previous elastic strain. It will further be seen that when the load is removed and re-applied during this range, flow again continues at the lower yield stress.

Table II shows that the mean value of the initial yield stress is greater in the longitudinal direction than in the transverse direction, for each material. The lower yield stress also, except in the case of steel A, is greater in the longitudinal direction and, in the case of steel B, by approximately the same amount. Steel A appears in fact to be more uniform in properties in the two directions that either B or C. The amount

TABLE II.—Results of Tension Tests.

Steel.	No. of Test-piece.	Direction of Axis of Test-piece.	Diameter.	Yield Stress. Initial.	Yield Stress. Lower.	Ratio, $\frac{\text{Lower Yield Stress}}{\text{Initial Yield Stress}} = \mu$.
			in.	ton/sq. in.	ton/sq. in.	
A	1	Longitudinal ...	0·2500	20·9	15·4	0·74
	2	" ...	0·2410	20·4	15·9	0·78
	3	" ...	0·3104	20·8	14·4	0·69
	4	" ...	0·3104	20·4	14·2	0·70
	5	" ...	0·3130	21·4	15·0	0·70
			Average ...	20·8	15·0	0·72
	6	Transverse ...	0·1870	20·45	15·3	0·75
	7	" ...	0·1853	19·2	15·4	0·80
	8	" ...	0·1867	18·8	15·1	0·80
			Average ...	19·5	15·3	0·78
B	1	Longitudinal ...	0·3110	24·7	16·9	0·68
	2	" ...	0·3124	25·6	17·0	0·66
	3	" ...	0·3094	25·6	17·3	0·67
			Average ...	25·3	17·1	0·67
	4	Transverse ...	0·1880	21·0	15·2	0·72
	5	" ...	0·1870	23·6	15·2	0·64
	6	" ...	0·1863	22·1	14·5	0·66
			Average ...	22·3	15·0	0·67
C	1	Longitudinal ...	0·3123	20·5	14·6	0·71
	2	" ...	0·3120	21·1	15·4	0·73
	3	" ...	0·3133	22·9	15·4	0·67
	4	" ...	0·3121	21·8	16·5	0·76
			Average ...	21·6	15·5	0·72
	5	Transverse ...	0·1874	19·2	14·8	0·77
	6	" ...	0·1875	18·9	14·6	0·77
	7	" ...	0·1879	18·5	14·9	0·81
			Average ...	18·9	14·8	0·78

of the drop in stress between the initial and lower yield points averages 25 per cent. for steels A and C, and 33 per cent. for steel B.

In the account which follows of the tests carried out in torsion, flexure and cylinders under pressure, theoretical treatments will first be given based upon the character of the stress-strain relation revealed by the tension test, as it was the inference from this theoretical investigation that determined some of the methods of test. It will be assumed that failure takes place when the shear stress reaches a critical value; that the stress then falls to, and remains at, a lower value during subsequent overstrain, and that this process has been followed by every portion suffering overstrain. Theoretical load-displacement diagrams will thus be obtained. By comparison with experimental diagrams the value of the lower yield stress will be inferred. The surface stress at the initial breakdown will be calculated by the usual methods.

4. *Torsion.*

The theoretical relation between the torque and angle of twist of an overstrained cylindrical bar, making the assumptions referred to above, was given in a previous paper.* It will, however, be convenient to recapitulate the steps of the analysis, and to express the result in a slightly different form. Referring to fig. 7, the stress distribution at the instant prior to elastic breakdown at the surface if represented by the triangle OAB, OA being the radius, and AB the shear stress s at the outer surface. On increasing the angle of twist the outer layers yield, the stress in them is assumed to fall to the constant lower yield stress s' represented by AC. When overstrain has proceeded to a depth AA′, there will be a cylindrical core, radius OA′, remaining elastic, the outer surface of which is on the point of yielding at the shear stress s. Surrounding this will be a shell of overstrained material, yielding plastically under uniform shear stress s'. The stress distribution at this point is accordingly represented by the figure OACC′B′O.

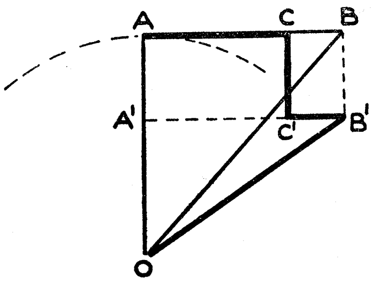

FIG. 7.

Denoting the radius OA by a, and OA′ by x, the twisting moment T corresponding to this stress distribution is readily seen, by integration, to be

$$T = \tfrac{1}{2}\pi s x^3 + \tfrac{2}{3}\pi s' (a^3 - x^3).$$

Let θ be the angle of twist per unit length. Then

$$s/x = C\theta.$$

where C is the modulus of rigidity.

* ROBERTSON and COOK, *loc. cit.*

If T_y, θ_y are the torque and twist when yield first occurs,

$$s/a = C\theta_y,$$

so that

$$x/a = \theta_y/\theta.$$

and therefore, since $T_y = \frac{1}{2}\pi s a^3$, we have

$$T = \tfrac{1}{2}\pi s a^3 \left(\frac{\theta_y}{\theta}\right)^3 + \tfrac{2}{3}\pi s' a^3 \left\{1 - \left(\frac{\theta_y}{\theta}\right)^3\right\},$$

$$= T_y \left(\frac{\theta_y}{\theta}\right)^3 + \tfrac{4}{3}\frac{T_y}{s} . s' \left\{1 - \left(\frac{\theta_y}{\theta}\right)^3\right\},$$

so that

$$\frac{T}{T_y} = \left(\frac{\theta_y}{\theta}\right)^3 + \tfrac{4}{3}\mu \left\{1 - \left(\frac{\theta_y}{\theta}\right)^3\right\}, \quad \ldots \ldots \ldots \quad (1)$$

where $\mu = s'/s$, the ratio of the lower to the initial yield stress.

By taking the ratios T/T_y, θ/θ_y as the variables, the theoretical torque-twist diagrams may be constructed without the necessity of defining the scales of torque and twist, and will be independent of the dimensions of the specimens. These are shown in fig. 10 by dotted lines for values of μ of 0·6, 0·7 and 0·8. It is seen from equation (1), or from the diagrams in fig. 10, that if the stress in the overstrained portion is less than 0·75 of the stress at the initial yield point at the surface, it will require less torque to continue the overstraining than to begin it. In torsion tests by dead load it is usually observed that at the yield point the twist continues without any increase in the torque. In some preliminary experiments on mild steel A, in which a pure torque was applied by dead weight, the suddenness and rapidity with which the twist increased suggested at once that the balance between the torsional resistance of the specimen and the applied torque had been destroyed, and that a torque less than that producing the initial yield was sufficient to continue the twist. Consequently it became necessary, in order to obtain a true torque-twist diagram, to adopt a method similar to that used in the tensile tests, namely, to subject the specimen to a steadily increasing strain which is unaffected by the induced torque, and to measure that torque by the elastic deformation of a weighbar. It would, of course, have been possible to have used a separate weighbar, twisted in series with the test-piece, for this purpose, as in the tension tests described above. But it was more convenient in this case, and made a more rigid arrangement, to use a portion of the test-piece itself as a weighbar. This, of course, is only possible where the material under test obeys Hooke's law very closely over, at any rate, a considerable range of stress. The specimens were accordingly machined to the shape and dimensions shown in fig. 3, being taken, as in the longitudinal tension specimens, from the portion of the rough bar between the centre and the surface. The parallel portion A, machined to a diameter of 0·23 inch, formed the weighbar. The portion B, machined to a diameter of 0·20 inch for a parallel length of $\frac{3}{8}$ inch and connected by fillets of large radius to the ends, formed the test-piece. The surface shear stress in these two portions will always be in the inverse ratio of the cube of the diameters, so that when the yield point is reached in the test portion the maximum stress in the weighbar will only be 0·66 of the yield stress and is thus far removed from any possibility of overstrain. Holders for mirrors were attached at points $1\frac{1}{16}$ inch apart on the weighbar, and at each end of the parallel part of the test portion, each pair being arranged so that the image of a scale was seen through a telescope by double reflection. Any rotation as a whole was thus automatically corrected, and a single reading gave a measure of the true angle of twist in each part. The mirrors of the weighbar were brought close to those of the test portion so that the scales appeared side by side in the field of view of a single telescope. The arrangement is shown diagrammatically in fig. 8. In order to infer the torque from the weighbar readings, the scale reading for a known torque applied by dead load was obtained at the commencement of each test.

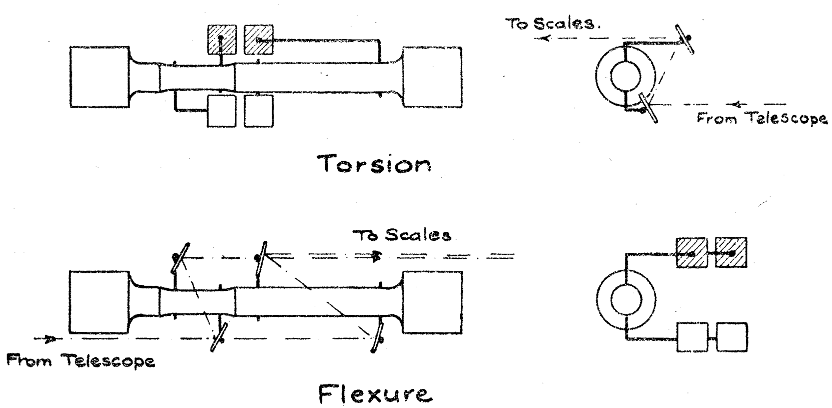

FIG. 8.—Arrangement for Load-Strain Measurement in Torsion and Flexure.

The arrangements for applying the torsional strain will be seen from fig. 9. One end of the specimen A (the weighbar end) was rigidly held in a fixed bracket. To the other end, which was quite free, was keyed a drum B attached to which were two steel tapes. One of these led downwards to one end of a horizontal bar C. The other passed over a drum D mounted on ball bearings, and was attached to the other end of the bar C. The pull of the straining mechanism was applied to the centre of this bar by a method identical with that described in the tension tests. A pure torque was thus obtained with the maximum of rigidity in the straining appliances. Room was provided for applying a dead load of 12 lb. to the centre of the bar for purposes of calibration.

The torsional strain was applied at a uniform rate up to the yield point, the time occupied being approximately the same as in tension. At this point there was a rapid diminution in torque, particularly in steels A and B, accompanied by a considerable increase in twist due to the relief of the straining appliances. The drop was not so sudden as in tension, but was too rapid to obtain corresponding readings of torque and twist until an additional twist of about 50 per cent. of the twist at the yield point had occurred. Three representative torque-twist diagrams are shown in fig. 10. The

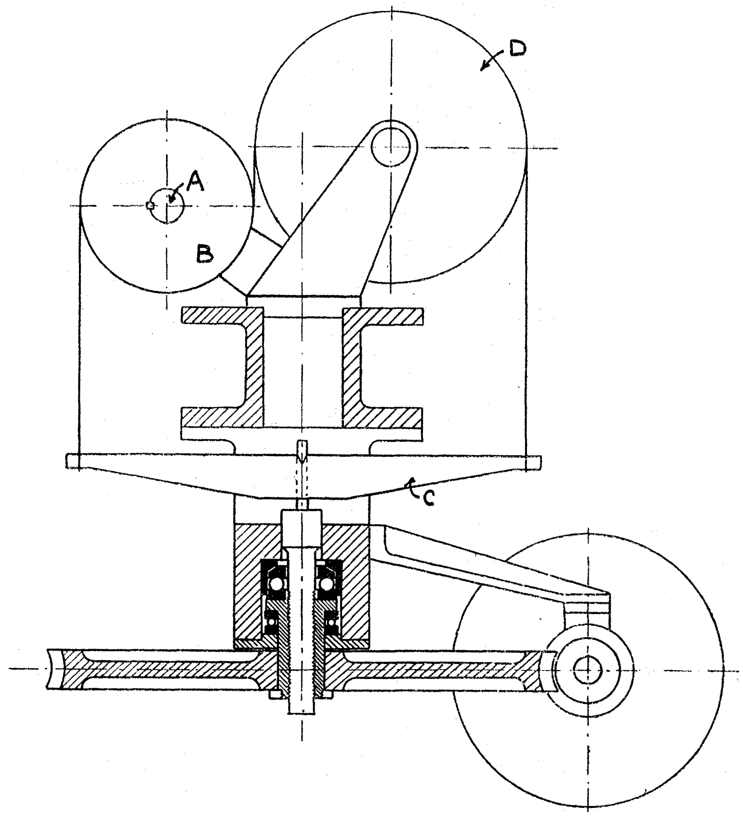

FIG. 9.—Apparatus for Straining in Torsion.

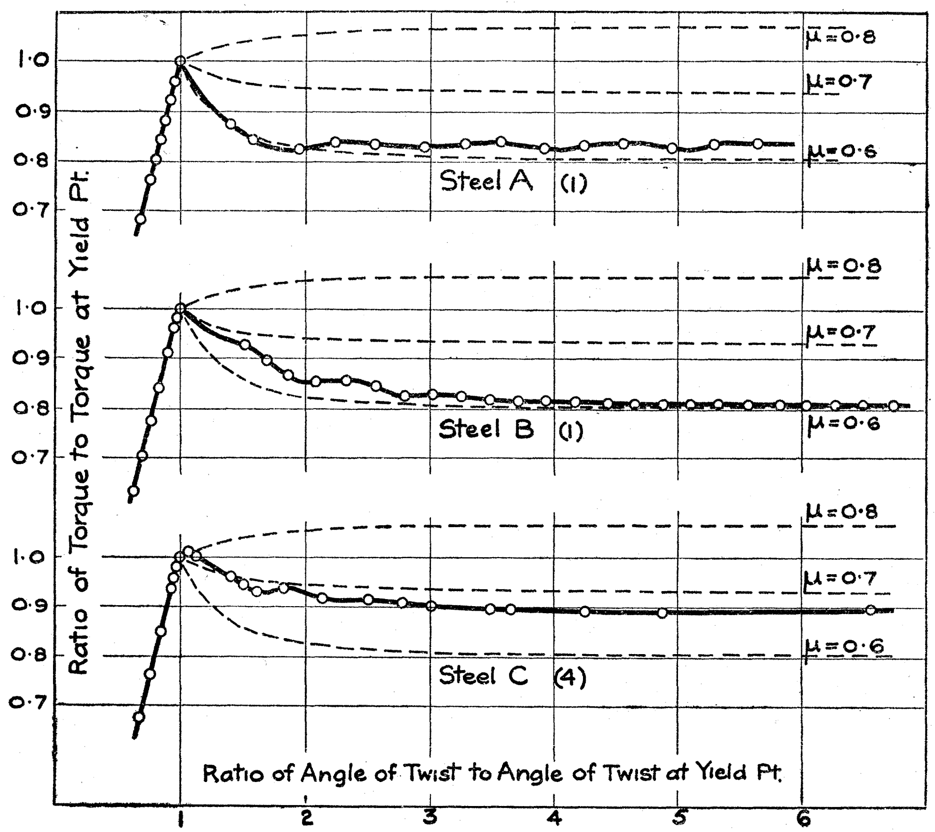

FIG. 10.—Torque-Twist Diagrams.

elastic line in each case showed no appreciable deviation from the straight and for reasons of space only the portion near the yield point is shown. After the yield point was passed the torque diminished, but tended to become approximately constant, with a value considerably less than that causing the yield, over a range of twist at least six or seven times the twist at the yield point. It will be seen at once that there is a strong resemblance between the theoretical and experimental diagrams, but it is desirable to mention here that this resemblance is not conclusive proof of the correctness of the process of failure assumed by the author. The angle of twist measured is the total twist over a definite length, and there is no guarantee that this will be uniformly distributed. The experimental diagram is, in fact, not inconsistent with a process of overstrain whereby each successive layer yields at a value of the shear stress rather less than that in the outer layer at the initial yield point, if the process does not take place simultaneously at every cross section. This point is of some importance, and will be alluded to later in the general discussion of the results. There is no uncertainty, however, about the value of the stress in the overstrained zone which may be inferred from these diagrams. For a comparatively large twist ratio the torque-twist line is approximately horizontal. The elastic core is then of small diameter, and contributes

Table III.—Results of Torsion Tests.

Steel.	Test-piece No.	Diameter.	Torque at Initial Yield T_y.	Shear Stress at Initial Yield s.	Torque T for $\theta/\theta_y = 5$.	Ratio T/T_y.	$\mu = \frac{3}{4}T/T_y$.	Inferred Shear Stress in Over-strained Layers $s' = \mu s$.
		in.	lb./ins.	tons/ sq. in.	lb./ins.			tons/ sq. in.
A	1	0·1983	43·7	12·7	36·3	0·83	0·62	7·9
	2	0·1983	41·3	12·1	36·4	0·88	0·66	8·0
	3	0·1995	43·1	12·4	37·1	0·86	0·645	8·0
	4	0·2003	43·7	12·4	37·6	0·86	0·645	8·0
			Average	12·4	—	—	0·64	8·0
B	1	0·2009	50·2	14·1	40·7	0·81	0·61	8·6
	2	0·2000	48·3	13·7	39·6	0·82	0·62	8·5
			Average	13·9	—	—	0·615	8·5
C	1	0·2005	41·6	11·8	35·8	0·86	0·645	7·6
	2	0·2009	41·6	11·7	37·0	0·89	0·67	7·8
	3	0·2006	45·3	12·8	36·7	0·81	0·61	7·8
	4	0·2000	42·4	12·0	37·7	0·89	0·67	8·0
	5	0·1970	38·2	11·4	34·4	0·90	0·675	7·7
			Average	11·9	—	—	0·65	7·8

very little to the moment of resistance, so that whatever may be the stress conditions in the core, the torque ratio will not be affected appreciably. Equation (1) shows that for large values of the twist ratio θ/θ_y

$$\mu = \tfrac{3}{4}\ T/T_y \text{ approximately.}$$

For a twist ratio of 5, the error involved is less than 0·2 per cent. The results of all the torsion tests are given in Table III. The shear stress s at the initial yield point is calculated from the usual formula

$$s = 2\ T_y/\pi a^3$$

and the remaining figures are explained at the head of each column. They will be discussed later in connection with the results of other tests.

5. *Flexure.*

In simple uniform flexure the stress at each point is one of simple tension or compression. There are many difficulties encountered in any attempt to get the true shape of the stress-strain diagram in compression. But the results of such investigations as have been made indicate a close similarity in the behaviour of ductile steels and iron as between tension and compression. The most reliable information available is that obtained by ROBERTSON,* and the results for three materials are given in the following table :—

TABLE IV.—Professor ROBERTSON'S Experiments.

Material.	Tension.			Compression.		
	Initial Y.P.	Lower Y.P.	Young's Modulus.	Initial Y.P.	Lower Y.P.	Young's Modulus.
	tons/sq. in.	tons/sq. in.	tons/sq. in.	tons/sq. in.	tons/sq. in.	tons/sq. in.
Mild steel	20·2	—	13,300	19·2	16·7	13,300
36-ton steel...	23·0	—	13,200	24·1	23·6†	13,500
Armco iron as received	9·4	7·5	12,750	9·6	7·3	13,130
Armco iron as received, normalised	8·3	7·3	—	10·0	7·7	—

† Estimated from diagram.

Unfortunately for the present purpose, ROBERTSON does not appear to have determined the lower yield point in tension for the two steels. The initial yield stresses in

* "The Strength of Struts," 'Inst.C.E.,' Selected Eng. Paper No. 28, 1925. "The Drop of Stress at Yield in Armco Iron,' 'Aeronautical Res. Com.,' R. & M., No. 1161.

tension and compression do not differ by a greater amount than will be found in successive tests in either tension or compression. The lower yield point in Armco iron is practically the same in tension and compression. The elastic stress-strain relation in compression was found to be linear for the steels, and the value of the modulus approximately the same as in tension.

It is a reasonable assumption therefore, at any rate as a first approximation, that for mild steel both initial and lower yield points are the same in tension and compression, and consequently that the stress distribution is the same on each side of the neutral surface. In the flexure tests to be described, the specimens used were those which had already been tested in torsion and subsequently re-normalised. A circular cross section only will therefore be considered. The tensile (and compressive) stress at the initial yield point will be denoted by f, and the lower yield stress by f'. Assuming a process of overstrain similar to that adopted for torsion, fig. 7 will again represent the stress distribution on one side of the neutral surface, the initial yield stress f being represented by AB, and the lower yield stress f' by AC. The moment of resistance M when the overstrain extends inwards through a distance AA′, where $OA' = x$, will be easily seen to be

$$M = \frac{4fa^4}{x}\int_0^\alpha \sin^2\lambda\cos^2\lambda\,d\lambda + 4f'a^3\int_\alpha^{\frac{1}{2}\pi}\sin\lambda\cos^2\lambda\,d\lambda,$$

where $\sin\alpha = x/a$, or,

$$M = \frac{fa^4}{2x}(2\sin^3\alpha\cos\alpha - \sin\alpha\cos\alpha + \alpha) + \tfrac{4}{3}f'a^3\cos^3\alpha.$$

It will be convenient to represent the flexural deformation by the curvature c. Considering the elastic portion, we have

$$f/x = Ec,$$

where E is YOUNG's Modulus. At the initial yield point,

$$f/a = Ec_y,$$

where c_y is the curvature when yield first takes place.

Hence,

$$x/a = c_y/c.$$

Also, since

$$M_y = f\,.\,\pi a^3/4, \quad \text{and} \quad f'/f = \mu,$$

$$M = \frac{2M_y}{\pi}\,.\,\frac{a}{x}(2\sin^3\alpha\cos\alpha - \sin\alpha\cos\alpha + \alpha) + \tfrac{16}{3}M_y\frac{\mu}{\pi}\,.\cos^3\alpha,$$

and

$$\frac{M}{M_y} = \frac{1}{\pi}\left[2\sqrt{1-\left(\frac{c_y}{c}\right)^2}\left\{2\left(\frac{c_y}{c}\right)^2 - 1\right\} + 2\frac{c}{c_y}\sin^{-1}\frac{c_y}{c} + \tfrac{16}{3}\mu\left\{1-\left(\frac{c_y}{c}\right)^2\right\}^{3/2}\right]. \quad . \quad . \quad (2)$$

This equation expresses the bending moment ratio M/M_y in terms of the curvature ratio c/c_y, and curves plotted from it are given in fig. 12 for values of μ of 0·5, 0·6 and 0·7.

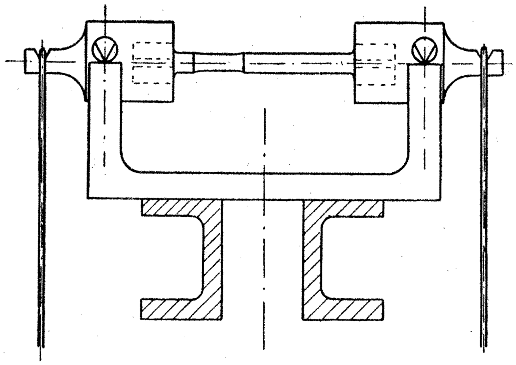

FIG. 11.—Bending Moment Curvature Diagrams for Flexure.

It was anticipated from the torsion tests, and from the theoretical curves shown in fig. 12 that, although the initial yield would not be accompanied by an immediate

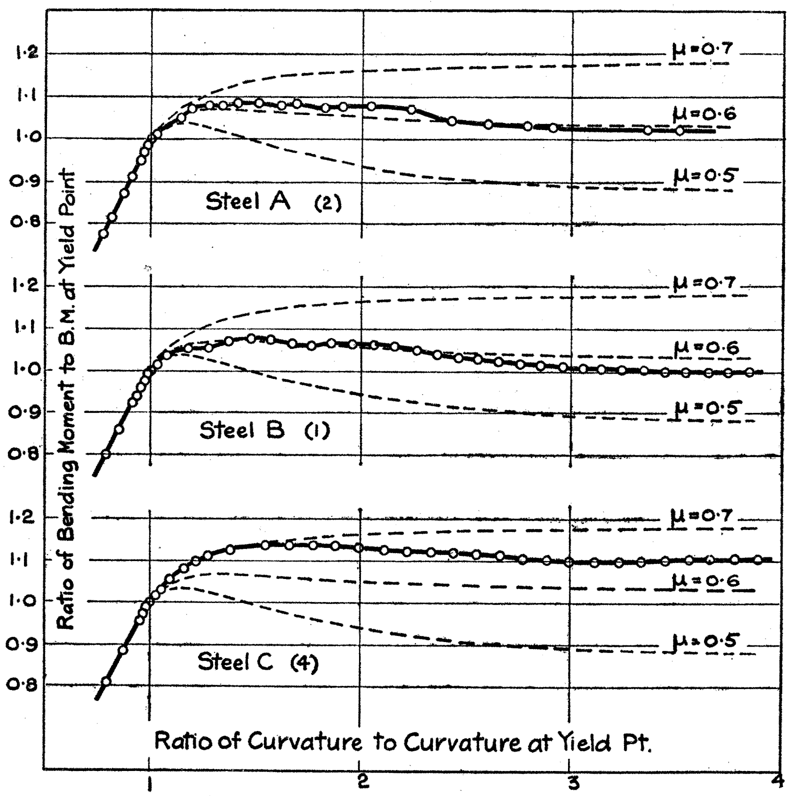

FIG. 12.—Apparatus for Straining in Flexure.

fall in bending moment analogous to the effect seen in tension and in torsion, the subsequent curve might not be a continuously rising one, and it was consequently decided that the method of test should be the same as that employed in torsion. The apparatus used in the torsion tests was very easily adapted for the flexure tests. The same straining mechanism was used. The cylindrical ends of the specimen fitted into holders provided with knife edges, the plane of the edges containing the axis of the specimen. The arrangement is shown in fig. 12. The bending strain was applied through vertical rods leading to the ends of the beam used in the torsion tests. By this means a uniform bending moment was applied. The fit of the specimen in the holders was sufficiently free to allow the knife edges to take up their position on the pedestals without producing any torsional strain in the specimen.

A slight modification to the mirror attachments used in torsion enabled them to be employed for measuring the change in curvature. The relative rotation of two sections, in the plane of flexure, observed by double reflection of the image of a scale, gave a measure of the change of curvature between the sections. From the curvature of the enlarged portion of the specimen the bending moment was inferred—a calibration by dead weight providing the scale. As in the torsion tests, the mirrors were so arranged that the scales for the weighbar and the test portion appeared side by side in the field of view of a single telescope, as shown diagrammatically in fig. 8. Three representative diagrams obtained in this way are shown in fig. 12.

It will be seen that there is again a strong resemblance between the bending moment-curvature diagrams obtained for these steels and the curves plotted from equation (2). After the yield point has been passed, the bending moment continues to increase for some time, but later gradually falls as the curvature increases. This is a characteristic of the theoretical curves for values of μ less than about $0 \cdot 65$. The same reservation must, however, be made, as in the torsion diagrams, regarding the inference to be drawn from this similarity. But we may again calculate with confidence the value of μ from the bending moment corresponding to the higher curvature ratios, where the curve becomes horizontal. Equation (2) shows that the value of μ is then approximately $3\pi/16 \, . \, M/M_y$. With a curvature ratio of 4, the error involved is less than 1 per cent.

Table V contains the complete results of the flexure tests. The tensile and compressive stress f at the initial yield point, given in the fifth column, is calculated from the usual formula

$$f = 4M_y/\pi a^3.$$

The bending moment ratio for a curvature-ratio of 4 is used for the calculation of μ, and the stress f' in the overstrained region is inferred.

These results will be discussed later, but attention may here be directed to the relation between the figures given in column 5 and those in the corresponding column of Table III. The shear stress at the yield point in torsion is very approximately one-half the tensile stress at the same point in flexure.

TABLE V.—Results of Flexure Tests.

Steel.	Test-piece No.	Diameter.	Bending Moment at Initial Yield M_y.	Maximum Tensile or Compressive Stress at Initial Yield f.	Bending Moment M for $c/c_y = 4$.	Ratio M/M_y.	$\mu = \frac{3\pi}{16} \cdot \frac{M}{M_y}$.	Inferred Tensile and Comp. stress f' in Over-strained Layers $= \mu f$.
		in.	lb./ins.	tons/sq. in.	lb./ins.			tons/sq. in.
A	1	0·1983	41·8	23·8	44·3	1·06	0·625	14·9
	2	0·1995	42·8	24·7	44·1	1·03	0·61	15·1
	3	0·2003	42·5	24·2	43·8	1·03	0·61	14·8
			Average	24·2			0·615	14·9
B	1	0·2009	50·2	28·1	50·2	1·000	0·59	16·6
	2	0·2000	49·9	28·4	50·0	1·002	0·59	16·8
			Average	28·2			0·59	16·7
C	1	0·2007	44·6	25·1	44·6	1·000	0·59	14·8
	2	0·2002	42·1	23·9	47·1	1·12	0·66	15·8
	3	0·2004	40·0	22·7	46·8	1·17	0·69	15·7
	4	0·2000	42·7	24·3	47·2	1·10	0·65	15·8
	5	0·1970	41·7	24·8	45·1	1·08	0·64	15·9
			Average	24·2			0·65	15·6

6. *Thick-walled Cylinders under Internal Hydrostatic Pressure.*

The stresses in a thick-walled cylinder of a perfectly elastic material subjected to internal pressure are completely known at every point except in the immediate neighbourhood of the ends. They may be calculated by a method, originally due to LAMÉ, which is given in most text-books. The only assumption made in the theory (other than that of perfect elasticity) is that the axial strain is uniform over the whole cross-section. It is clear, from a consideration of the consequences which any other distribution of strain would have in a long cylinder, that the assumption is valid.

It is shown by this theory that the three principal stresses p, q, r in the radial, circumferential, and axial directions respectively at any radius a are given by

$$p = A + B/a^2,$$
$$q = A - B/a^2,$$
$$r = C.$$

where A, B and C are constants determinable when the boundary conditions are known. The stresses p and q will have their maximum values at the internal surface. Denoting the internal and external radii by a_0 and a_1, and the internal pressure by P, we have at the internal surface of a cylinder with closed ends (so that the axial stress is due to the internal pressure only)

$$p_0 = \mathrm{P},$$

$$q_0 = -\mathrm{P} \cdot \frac{a_1^2 + a_0^2}{a_1^2 - a_0^2} = -\mathrm{P} \cdot \frac{k^2 + 1}{k^2 - 1},$$

$$r_0 = -\mathrm{P} \cdot \frac{a_0^2}{a_1^2 - a_0^2} = -\mathrm{P} \cdot \frac{1}{k^2 - 1},$$

where $k = a/a_0$, and compressive stresses are taken as positive. Since p_0 and q_0 are both numerically greater than r_0, and are of opposite sign, they form the components of the maximum shear stress, which will be equal to

$$\tfrac{1}{2}(p_0 - q_0) = \mathrm{P} \,.\, k^2/(k^2 - 1).$$

The maximum strain energy, per unit volume, is given by

$$\frac{1}{2\mathrm{E}} \cdot \left\{p_0^2 + q_0^2 + r_0^2 - \frac{2}{m}(p_0 q_0 + q_0 r_0 + r_0 p_0)\right\} = \frac{\mathrm{P}^2}{2\mathrm{E}} \cdot \frac{2k^4 (1 + 1/m) + 3(1 - 2/m)}{(k^2 - 1)^2},$$

$1/m$ being POISSON's ratio.

As the internal pressure is increased, elastic breakdown will first occur at the internal surface, and will gradually extend outwards. The results of the previous tests* showed that the maximum shear stress at the yield point was about 20 per cent. greater than that observed in a tensile test. Although it has been pointed out by HAIGH that the results were in close agreement with the strain-energy theory of failure—they have, indeed, formed one of the strongest arguments in favour of that theory—the author is not inclined, for reasons given later, to discard the view that elastic breakdown takes place at a definite value of the shear stress. It will, at all events, be assumed in the theoretical treatment of the initial stages of overstrain that failure is by shear.

The distribution of stress in a cylinder overstrained by internal pressure has been investigated theoretically by TURNER (*loc. cit*) and both theoretically and experimentally by MACRAE.†

The assumption made by TURNER was that the material remained perfectly elastic up to the yield point, and that plastic flow then took place at a constant shear stress equal to the initial yield stress. The same assumption is made by MACRAE, with the modification that the range of strain at constant stress is limited, and is followed by a uniformly increasing stress, described as a "semi-plastic" state. This is probably a sufficiently close approximation for the gun steels considered. No difficulty is encountered in obtaining expressions for the shear stress at any point in the

* COOK and ROBERTSON, *loc. cit.*

† "The Overstrain of Metals, and its application to the Autofrettage Process of Cylinder and Gun Construction," H.M. Stationery Office, 1930.

cylinder wall for any of the above assumptions provided that shear stress is regarded as the determining factor both for initial failure and subsequent flow. There is, however, no means of verifying experimentally the theoretical conclusions other than by the measurement of strains. Agreement between the measured and theoretical diametral extension, for instance, would be strong evidence of the existence of the stress distribution assumed. Unfortunately, a difficulty is introduced in the calculation of the strains by the fact, to which attention was drawn by TURNER, that although in an overstrained cylinder axial strain will remain constant over the cross-section, it does not follow, as in the case of a completely elastic cylinder, that the axial stress will also be constant. If the cylinder is partially overstrained the axial strain must necessarily be uniform over the outer portion, which remains elastic. Its magnitude, however, depends on that in the overstrained portion. TURNER has shown how, by making assumptions regarding the nature of plastic strain, a differential equation may be obtained involving the axial stress, but has not attempted a solution, and it does not appear that any simple solution exists.

Under these circumstances it is proposed to make the simple supposition that the axial stress does not differ materially from the value it would have under a uniform distribution. It is fully recognised that results thus obtained must be considered as a first approximation only, but provided that the thickness of the overstrained zone is not too great a proportion of the total thickness, the error is not likely to be considerable.

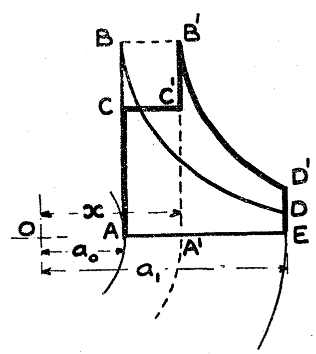

FIG. 13.

The process of overstrain assumed in the following analysis will be the same as that adopted in torsion and flexure. Referring to fig. 13, the distribution of shear stress, when the internal surface is on the point of elastic breakdown, is represented by the figure ABDE, where AB is equal to s, the shear stress at the yield point. The occurrence of yield is accompanied by a drop in stress to the lower value s', represented by AC, and remains constant at this value during the subsequent overstrain. When, therefore, the overstrain has extended to a point A′ we shall, on this assumption, have a region between A and A′ in which plastic flow is taking place under constant shear stress s'. The portion outside this, being still elastic, will have the ordinary stress distribution for a thick cylinder, and since the inner surface of this shell is on the point of yielding, the shear stress there will be s. The figure ACC′B′D′E will now represent the stress distribution.

In the inner (overstrained) zone, if p and q are the radial and circumferential stresses at radius a (compressive stresses being taken as positive), we have, considering the equilibrium of an element of a cylindrical shell bounded by radial planes,

$$a\frac{dp}{da} = q - p.$$

Since the maximum shear stress is one-half the difference of the two relevant principal stresses, and is constant and equal to s', we have

$$p - q = 2s'.$$

Hence

$$a\,\frac{dp}{da} = -\,2s',$$

and

$$p = -\,2s' \log_e a + \mathrm{C},$$

where C is a constant. At the internal surface $p = \mathrm{P}$, the internal hydrostatic pressure, so that

$$\mathrm{P} = -\,2s' \log_e a_0 + \mathrm{C},$$

and therefore

$$p = 2s' \log_e (a_0/a) + \mathrm{P}.$$

Denoting the outer radius of the overstrained zone by x we have, for the radial stress at this point,

$$p_x = 2s' \log_e (a_0/x) + \mathrm{P} = \mathrm{P} - s' \log_e n^2,$$

where $n = x/a_0$. p_x may be regarded as the internal pressure acting at the internal surface of an elastic cylinder whose external and internal radii are a_1 and x. The circumferential stress at the inner surface of this cylinder is, by the usual theory,

$$q_x = -\,\frac{a_1^2 + x^2}{a_1^2 - x^2} \cdot p_x = -\,\frac{k^2 + n^2}{k^2 - n^2} \cdot p_x,$$

k, as before, being the ratio a_1/a_0. Since this surface is on the point of yielding, the stress there is s. Consequently,

$$\begin{aligned} s &= \tfrac{1}{2}\,(p_x - q_x) \\ &= \tfrac{1}{2}\left(1 + \frac{k^2 + n^2}{k^2 - n^2}\right) p_x \\ &= \frac{k^2}{k^2 - n^2}\,(\mathrm{P} - s' \log_e n^2). \qquad (3) \end{aligned}$$

Now let P_y be the internal pressure producing the initial yield. Putting $n = 1$, we have

$$s = \frac{k^2}{k^2 - 1} \cdot \mathrm{P}_y,$$

so that

$$\begin{aligned} \frac{k^2}{k^2 - 1} \cdot \mathrm{P}_y &= \frac{k^2}{k^2 - n^2}\,(\mathrm{P} - s' \log_e n^2) \\ &= \frac{k^2}{k^2 - n^2}\left(\mathrm{P} - \frac{s'}{s} \cdot \frac{k^2}{k^2 - 1} \cdot \mathrm{P}_y \log_e n^2\right) \qquad (4) \end{aligned}$$

whence

$$\frac{P}{P_y} = \frac{k^2 - n^2}{k^2 - 1} + \mu \cdot \frac{k^2}{k^2 - 1} \cdot \log_e n^2, \quad \ldots \ldots \ldots \quad (5)$$

denoting, as before, the ratio s'/s by μ.

Equation (5) gives the ratio of the pressure at any instant to the yield pressure in terms of μ and the ratio n of the radii of the boundaries of the overstrained zone. The ratio n cannot be directly measured. The radial extension of the outer surface can, however, be accurately measured by means of a suitable extensometer, so that if we can obtain a relation between n and this extension, it is then possible to express the pressure ratio in terms of the extension, and a comparison can be made with experiment.

Considering the portion remaining elastic we have, for the circumferential stress at the outer surface where the radius is a_1,

$$q_1 = -p_x \cdot \frac{2x^2}{a_1^2 - x^2} = -\frac{2n^2}{k^2 - n^2} \cdot p_x.$$

The axial stress r, due to the internal pressure acting on the closed ends of the cylinder, will, for reasons set out above, be assumed to be uniformly distributed over the whole cross-section, so that

$$r = -P \cdot \frac{a_0^2}{a_1^2 - a_0^2} = -\frac{P}{k^2 - 1}.$$

If the radial extension per unit radius at the outer surface is denoted by e, we have,

$$\begin{aligned} e &= -\frac{q_1}{E} + \frac{1}{m} \cdot \frac{r}{E} \\ &= \frac{p_x}{E} \cdot \frac{2n^2}{k^2 - n^2} - \frac{P}{mE} \cdot \frac{1}{k^2 - 1} \\ &= \frac{1}{E}\left\{(P - s' \log_e n^2) \cdot \frac{2n^2}{k^2 - n^2} - \frac{P}{m} \cdot \frac{1}{k^2 - 1}\right\}, \end{aligned}$$

$1/m$ being Poisson's ratio. Denoting by e_y the corresponding extension at the initial yield point, *i.e.*, when $n = 1$,

$$e_y = \frac{1}{E} \cdot \frac{P_y}{k^2 - 1}\left(2 - \frac{1}{m}\right),$$

whence

$$\frac{e}{e_y} = \frac{m}{2m - 1}\left(\frac{P - s' \log_e n^2}{P_y} \cdot 2n^2 \cdot \frac{k^2 - 1}{k^2 - n^2} - \frac{1}{m} \cdot \frac{P}{P_y}\right).$$

Now from (4),

$$P - s' \log_e n^2 = \frac{k^2 - n^2}{k^2 - 1} \cdot P_y,$$

therefore

$$\frac{e}{e_y} = \frac{m}{2m - 1}\left(2n^2 - \frac{1}{m} \cdot \frac{P}{P_y}\right) \quad \ldots \ldots \ldots \ldots \quad (6)$$

It is not possible to eliminate n directly from equations (5) and (6), but by giving successive values between 1 and k, between which it must lie, we obtain corresponding values of P/P_y and e/e_y, and curves can be plotted showing the relation between these quantities for various values of μ. These curves are shown by broken lines in fig. 14 for cylinders for which $k = 3$. A determination of POISSON'S ratio for steel A gave the value 0·29*, and this has been used in all the calculations. The curves may be regarded as load-displacement diagrams for the thick cylinder analogous to those obtained for torsion and flexure. The use of ratios is again particularly convenient for the comparison with experimental results, as it avoids the necessity of introducing elastic constants (other than POISSON'S ratio) and conversion factors for the extensometer readings.

Initial Yield Point in Cylinders.—The previous tests had for their object the determination of the stress relations at the yield point, and the actual bursting pressure.

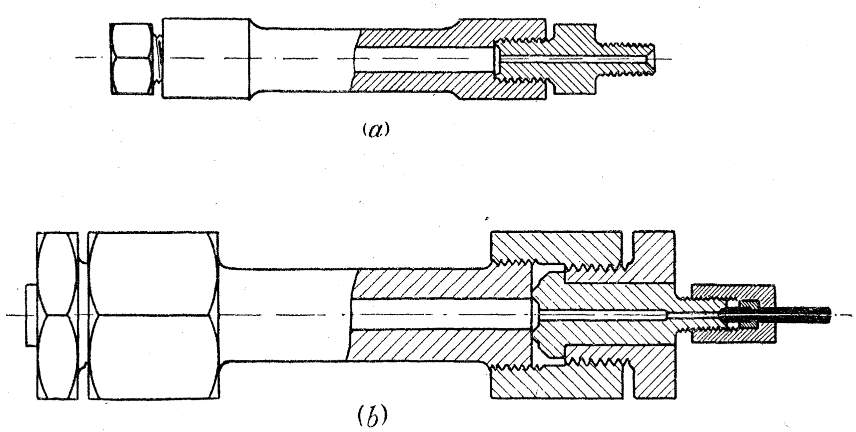

FIG. 14.—Test-pieces and Pressure Connections for Thick Cylinder Test.

They were made on cylinders bored to $\frac{5}{8}$ inch internal diameter, having different wall thicknesses, and no attempt was made to investigate the stress distribution across the cylinder wall after the yield point had been passed. In the present tests the first group of cylinders, nine in number, was prepared from the mild steel A, with a constant internal diameter of $\frac{5}{16}$ inch, and a ratio of external to internal diameter varying from 3·0 to 1·4. Particulars of the dimensions of these cylinders are given in Table VI under Nos. A9 to A1. They were very accurately bored and reamered, machined externally to the required dimensions, and then normalised *in vacuo*. Fig. 14 (*a*) shows the general shape, together with the type of pressure connection used. It was known from special tests carried out in the previous investigation that for this size of cylinder a length between fillets of $2\frac{1}{4}$ inches would be amply sufficient to prevent the enlarged ends having any strengthening effect on the central portion.

The high-pressure pump used in the previous tests was no longer available; it was

* This value is confirmed by the experiments of SMITH and COX on similar material. 'Aeronautical Research Committee,' R. & M., No. 1138 (1927).

not designed for the particular purpose for which it was employed, and proved inconvenient in use. For the present investigation the author accordingly designed the apparatus shown in fig. 15. It consists of a forging of high-tensile steel bored at one end for about half its length for a ram $\frac{3}{8}$ inch diameter, which was worked by a screw and large handwheel. To the inner end of the hole connections were provided (1) to a low-pressure pump, which could be shut off by a needle valve, and (2) to the pressure gauge and cylinder under test. The stuffing box for the ram was packed with about ten rings of rubber-cored packing dusted with graphite and driven well home. Using pure glycerine as the fluid it was found possible to produce pressures as high as 24 tons per square inch without leakage. The maximum pressure used in the present tests was about 18 tons per square inch, and this could be maintained

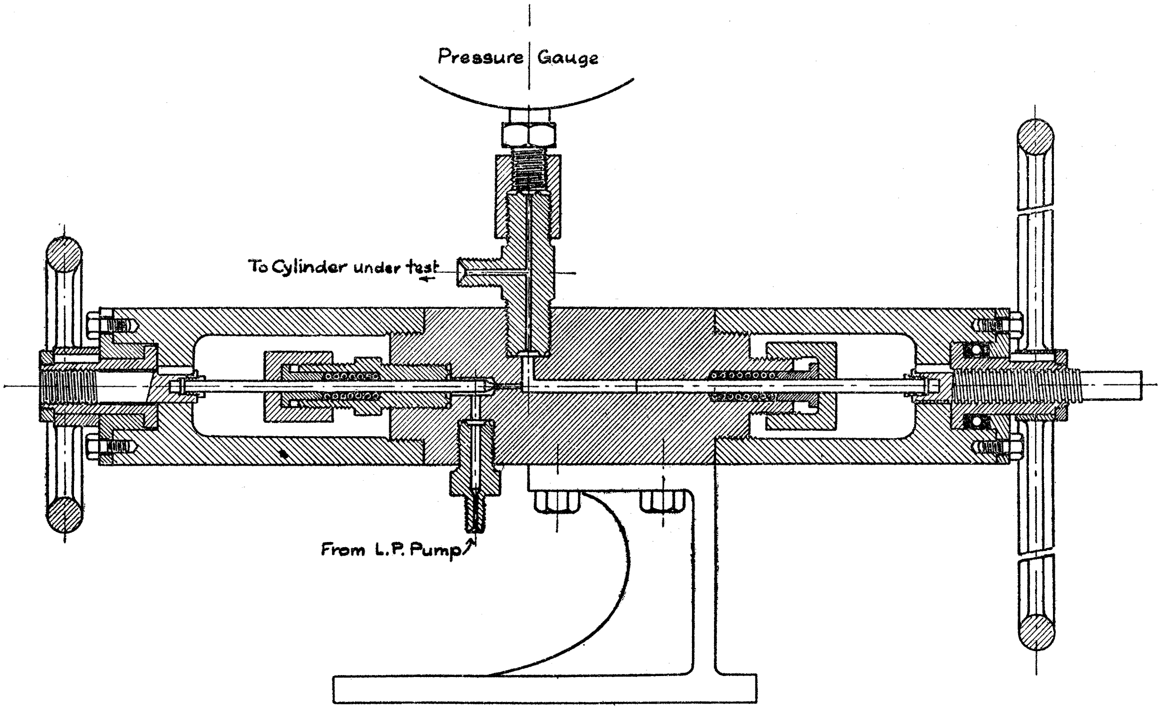

FIG. 15.—Intensifier for Production of High Pressures for Cylinder Test.

indefinitely without further advancing the ram. The pressure gauges were specially constructed by the Budenberg Gauge Co., and had ranges 0 to 15 tons per square inch and 0 to 20 tons per square inch. They were provided with jewelled bearings for the mechanism, and were carefully calibrated by the makers. After the 15-ton gauge had been in use for a considerable time it was tested against the new 20-ton gauge. A difference not greater than 0·05 tons per square inch was observed for pressures up to 6 tons per square inch, but above this pressure there was no measurable difference between the readings. The connection between the pressure apparatus and the cylinder under test was by means of small bore steel tubing about 5 feet long.

It was the intention, in this first group of cylinders, to repeat in the first place the tests carried out in 1911, namely, to measure the pressure required to cause elastic

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breakdown at the internal surface, and the method employed was very similar. A lateral extensometer was designed for the purpose of measuring the extension across a single diameter; this proved extremely satisfactory, and was sufficiently sensitive to indicate a change in diameter of the order of one-millionth of an inch. The principle was identical with that of the double extensometer shown in fig. 18, to be described later, and by which it was replaced in the later tests. The pressure was raised by intervals of 0·1 ton per square inch when approaching the yield point, and the test was discontinued at the first indication of a departure from a linear relation between pressure and extension. The pressure being then immediately released, no appreciable change took place in the dimensions of the cylinder which, when re-normalised, was available for further tests. Two or three tests were made on each cylinder in this way. The mean yield pressure, from which the separate values did not differ by more than 0·1 ton per square inch, is given for each cylinder in column 8 of Table VI. The shear stress, and also the strain energy per unit volume, at the internal surface are given in columns 10 and 11.

In order to compare the results with those obtained in the previous investigation, the ratio of the pressure at the yield point P_y to the transverse tensile yield stress f_t is given in column 9, and plotted as curve II in fig. 16. It will be seen that the curves

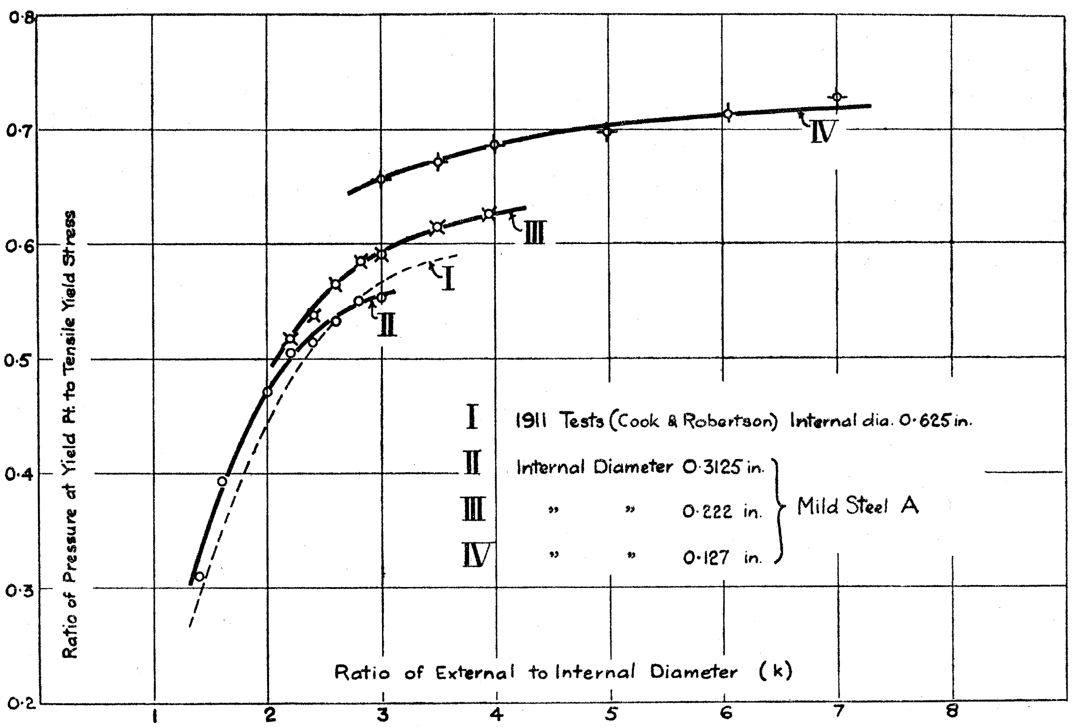

FIG. 16.—Comparison of Tests on Thick Cylinder having Different Internal Diameters.

I and II do not coincide. The materials, it is true, were not the same, but they were both mild steel, in which presumably the same law of failure would hold. For a given diameter ratio therefore the value of P_y/f_t should be the same, irrespective of the actual size of the cylinders. It appeared that the discrepancy might be due to errors in the determination of the transverse yield stress, either in the previous or in the present tests,* or else it was associated in some way with the size of the hole, which was $\frac{5}{8}$ inch diameter in the former tests and $\frac{5}{16}$ inch in the present. To settle this point two more cylinders of steel A were prepared, having internal diameters of approximately $\frac{7}{32}$ inch and $\frac{1}{8}$ inch, the external diameter in each case being $\frac{7}{8}$ inch. These were tested for the yield pressure in the manner already described. The ratio of external to internal diameter was then successively reduced by machining the outside, the cylinders being normalised and tested at each stage. The dimensions, and the results, are given in Table VI under numbers A 10 (*a*) to A 10 (*g*) and A 11 (*a*) to A 11 (*f*). Curves III and IV (fig. 17) have been plotted from them. It was not possible to use the extensometer with very small external diameters, and consequently the range of diameter ratios differs in each case, but the curves show very clearly that there is a scale effect, the ratio P_y/f_t increasing as the internal diameter is reduced. Selecting a diameter ratio of 3 : 1 the relation of P_y/f_t to the internal diameter is shown by the upper curve in fig. 17. The specimens

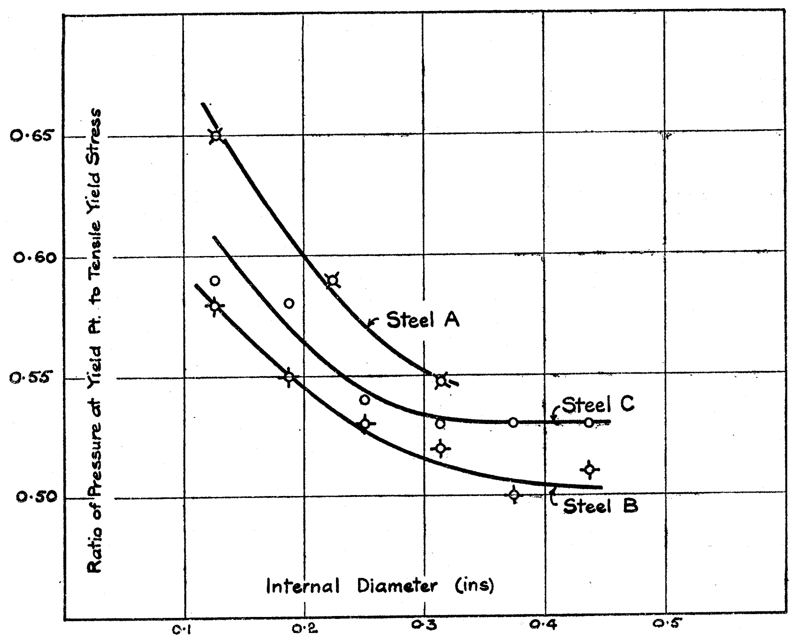

FIG. 17.—Scale Effect observed at Internal Yield Point in Cylinders having ratio, External to Internal Diameter, of 3.

used in these tests were machined from adjacent portions of the same bar, and it is therefore legitimate to exclude any uncertainty regarding the transverse yield stress in considering the differences obtained. The shear stress at the internal surface is seen from Table VI to be almost constant for the same internal diameter, but to increase as the diameter is reduced.

A different scheme was adopted in the specimens prepared from steels B and C. A set of six cylinders was machined in each case having the same diameter ratio of 3 : 1

* The specimens used in the 1911 tests were not all cut from the same bar.

throughout, the internal diameter varying from $\frac{1}{8}$ inch to $\frac{7}{16}$ inch, and in order to approach the conditions of similarity as nearly as possible, the length was made proportional to the other dimensions, except in the smallest, which would have been too short to accommodate the extensometer. The wide range of internal diameter made a modification of the type of pressure connection desirable, and the arrangement shown in fig. 14 (*b*) was accordingly substituted. The screwed ends of the cylinders were made to standard sizes appropriate to the cylinder dimensions, and the machining of the specimens was much simplified.

The dimensions and test results are given in the lower part of Table VI, and the curves in fig. 17 show the effect of variation in size. It is very marked in the case of steel B, but somewhat less so in steel C, which has a rather higher carbon content. With these cylinders the test was not broken off when the yield point was reached, but was continued into the plastic state, as will be described in the next section.

The results given in Table VI do not provide any very decisive answer to the question as to whether the maximum shear stress or the maximum strain energy is the true criterion of failure. The value of the strain energy is tabulated alongside that of the shear stress, and although in each series of tests the variation of strain energy is seen to be greater than that of shear stress, it is to be remembered that the former embodies the square of the internal pressure. For instance, in the first series A 9 to A 1, the mean variation from the average shear stress is 1 per cent., while that of the strain energy is 5 per cent. Taking the square root of the strain energy as providing a more legitimate comparison, the mean variation would be $2\frac{1}{2}$ per cent., which cannot be said to be wholly outside the limits of experimental error. In the following two series the degree of constancy is about the same for both. They are both apparently affected by the size of the hole.

Overstrain in Cylinders.—So long as the strain is elastic, the radial extension of a thick cylinder is symmetrical about the axis, and measurements on a single diameter only are necessary. After the yield point has been reached at the internal surface the conditions are entirely altered. In the previous experiments it was observed that at this point the diameter across which the extensometer was placed began in some cases to decrease instead of increase. A similar effect had been noticed by TURNER.* The explanation suggested in a later paper is probably the true one.† The yield would not occur simultaneously over the whole internal surface. It would begin at a single point, or possibly at opposite ends of a diameter, and the stress at these points would immediately fall to the lower yield stress. The effect upon the remaining elastic portion of the cylinder would thus be somewhat similar to that produced by a hole slightly elliptical in shape. Internal pressure would now cause unequal extensions of the two diameters, the diameter in the direction of the minor axis increasing more than that of the major, and similar effects will be observed at the external surface. If the ellipticity

* 'Engineering,' vol. 92, p. 115 (1911).

† ROBERTSON and COOK, *loc. cit.*

TABLE VI.—Results of tests of thick-walled cylinders by internal pressure.

Steel.	No. of Specimen.	Dimensions.				Transverse Yield Stress in Tension f_t.	Conditions at Yield Point.				Conditions during Overstrain.	
		Int. Diam.	Ext. Diam.	Length	Ratio Ext. to Int. Diam. k.		Int. Press. P_y.	P_y/f_t.	Max. Shear Stress.	Max. Strain Energy.	μ.	Shear Stress s'.
		in.	in.	in.		ton/sq. in.	ton/sq. in.		ton/sq. in.	inch-tons/cu. in.		ton/sq. in.
A	A 9	0·312	0·937	2·25	3·00	19·5	10·70	0·550	12·0	0·0142	0·57	6·8
	A 8	0·312	0·875		2·80		10·65	0·545	12·2	0·0143	0·54	6·6
	A 7	0·312	0·812		2·60		10·35	0·530	12·1	0·0144	0·55	6·7
	A 6	0·312	0·750		2·40		10·00	0·515	12·1	0·0143		
	A 5	0·312	0·687		2·20		9·80	0·505	12·4	0·0152	0·57	7·1
	A 4	0·312	0·625		2·00		9·15	0·470	12·2	0·0146		
	A 3	0·312	0·562		1·80		8·55	0·440	12·4	0·0158		
	A 2	0·312	0·500		1·60		7·55	0·385	12·4	0·0162		
	A 1	0·312	0·437		1·40		6·00	0·310	12·2	0·0163		
	A 10(a)	0·222	0·875	2·25	3·94	19·5	12·20	0·625	13·0	0·0166		
	A 10(b)	0·222	0·777		3·50		12·00	0·615	13·1	0·0167		
	A 10(c)	0·222	0·665		3·00		11·50	0·590	12·9	0·0164		
	A 10(d)	0·222	0·625		2·82		11·40	0·585	13·0	0·0167		
	A 10(e)	0·222	0·577		2·60		11·00	0·565	12·9	0·0164		
	A 10(f)	0·222	0·532		2·40		10·50	0·540	12·7	0·0159		
	A 10(g)	0·222	0·488		2·20		10·10	0·520	12·7	0·0161		
	A 11(a)	0·127	0·890	2·25	7·00	19·5	14·20	0·730	14·5	0·0204		
	A 11(b)	0·127	0·767		6·04		13·90	0·715	14·3	0·0199		
	A 11(c)	0·127	0·630		4·97		13·60	0·695	14·2	0·0195		
	A 11(d)	0·127	0·508		4·00		13·40	0·685	14·3	0·0200		
	A 11(e)	0·127	0·445		3·50		13·10	0·670	14·3	0·0199		
	A 11(f)	0·127	0·388		3·00		12·80	0·655	14·4	0·0204		
B	B 1	0·125	0·375	1·69	3·00	22·3	12·90	0·580	14·5	0·0203	0·47	6·8
	B 2	0·187	0·562	1·69	3·00		12·20	0·545	13·8	0·0182	0·50	6·9
	B 3	0·250	0·750	2·25	3·00		11·80	0·530	13·3	0·0171	0·53	7·0
	B 4	0·312	0·937	2·81	3·00		11·55	0·520	13·0	0·0163	0·57	7·4
	B 5	0·375	1·125	3·37	3·00		11·20	0·500	12·6	0·0153	0·60	7·5
	B 6	0·437	1·312	3·94	3·00		11·30	0·505	12·7	0·0156	0·58	7·4
C	C 1	0·125	0·375	1·69	3·00	18·9	11·20	0·590	12·6	0·0153	0·50	6·3
	C 2	0·187	0·562	1·69	3·00		10·90	0·575	12·3	0·0145	0·57	7·0
	C 3	0·250	0·750	2·25	3·00		10·30	0·545	11·6	0·0129	0·62	7·2
	C 4	0·312	0·937	2·81	3·00		10·10	0·535	11·4	0·0124	0·62	7·1
	C 5	0·375	1·125	3·37	3·00		10·10	0·535	11·4	0·0124	0·62	7·1
	C 6	0·437	1·312	3·94	3·00		10·10	0·535	11·4	0·0124	0·61	7·0

is sufficient, the latter diameter may even diminish. As the overstrain proceeds, a variation in thickness of the overstrained zone will produce a similar effect. It is probable that a very slight variation in this thickness is sufficient to produce a marked effect on the elastic extension at the external surface. In the case of a thin elliptical tube of small eccentricity, it may be shown* that the major axis will remain stationary for an increase of internal pressure provided that

$$\frac{\delta}{d} = \frac{m^2}{m^2 - 1}\left(\frac{t}{d}\right)^2,$$

where d is the average diameter, t is the thickness, and δ is the initial difference between the lengths of the major and minor axes. It will thus be seen that an extremely small difference in the two diameters is sufficient, in a thin tube, to give a large proportionate difference in the extensions measured over these diameters. Although the analogy cannot be pressed, it is reasonable to infer that, in the thicker tubes of these tests, wide variations in the radial extension may be produced by comparatively small variations in the thickness of the overstrained zone, and therefore that the existence of such irregular extension does not denote any very pronounced lack of symmetry in the process of yielding. It does, however, render the measurement of the extension across a single diameter (which is sufficient for the determination of the yield point) valueless as an indication of the stress distribution when the yield point is passed. For this purpose the true circumferential strain is required. It would be given accurately by the mean of the extensions measured across a large number of different diameters at the same cross-section, but an instrument designed for this purpose would be extremely complicated if it could be made at all. There is, however, little reason to suppose that the mean of two measurements, on diameters at right angles to each other, would differ from the true mean by more than a negligible amount. The instrument shown in fig. 18 was accordingly designed by the author. It consists of a pair of lateral extensometers so arranged that the levers of one can pass through those of the other, enabling them to be located at the same cross-section of the cylinder. In each extensometer a light frame A carries a flat adjustable contact stud B which is held against the cylinder at one end of the diameter. It carries also the V's for the knife edges of the lever E, and a seating for the pivots of the rocker D. The lever E carries a fixed chisel-edged stud held against the other end of the diameter. A spring maintains the requisite pressure on the studs to enable the instrument to be held by friction when the cylinder is vertical, any tendency to rotate about the studs being prevented by stops attached to the frame, and not shown in the figure. The extension of the cylinder causes the lever E to rotate about the knife edges. The movement of the end of this lever is measured by the rotation it produces in a small rocker D carrying a mirror M_1. The rocker has three hardened steel points in the same plane, two of which rest on a seating on the frame, and the third against a light bar F attached flexibly to the end of the lever E. These points are adjustable so that the rocker radius can be made as small

* By a method similar to that due to R. V. SOUTHWELL for the critical pressure of a tube exposed to external pressure, given in MORLEY'S "Strength of Materials," 7th edition, p. 340.

as desired, and the magnification of the lever movement correspondingly increased It was found that for these tests a distance of about 0·05 inches between the centre-point and the line joining the other two provided sufficient sensitiveness without impairing the stability of the rocker. The rotation of the mirror M_1 was observed by

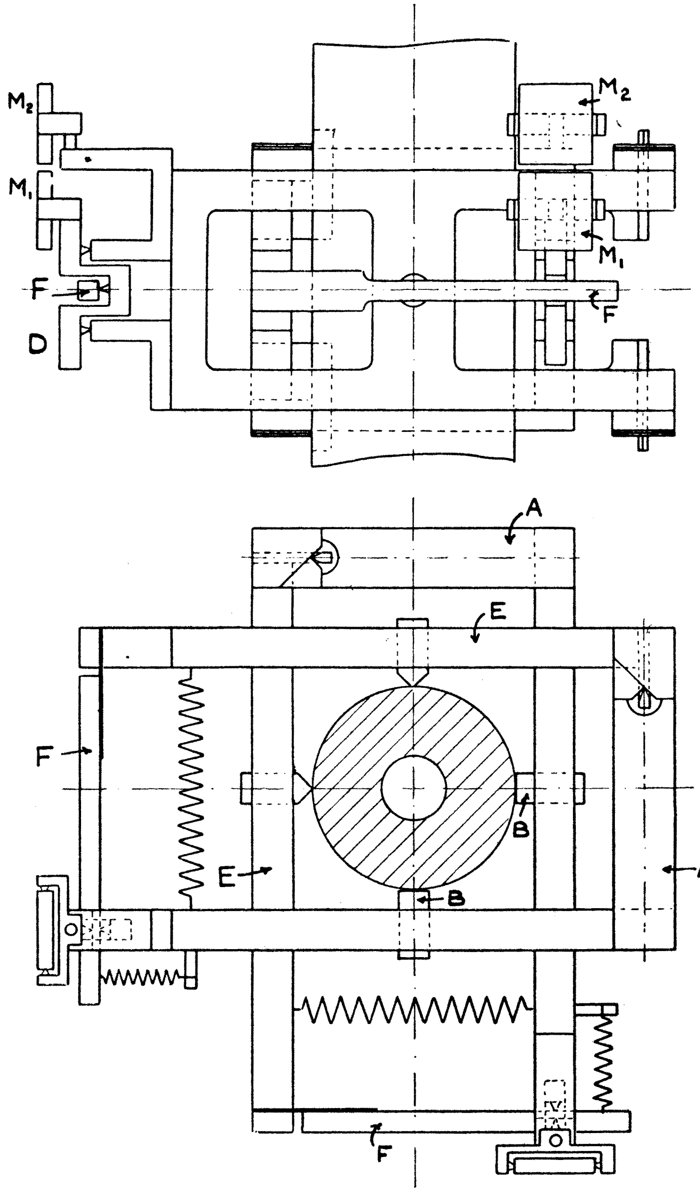

Fig. 18.—Double Extensometer for Measurement of Radial Extension in Thick Cylinders.

means of a telescope and scale, any displacement of the frame itself (which rarely occurred) being indicated by a second mirror M_2 attached to the frame, reflecting the same scale, and observed in the same field of view. At the scale distance usually employed, one scale division corresponded to an extension of approximately 6×10^{-6} inches, and it was possible to estimate to 0·1 of a scale division. Between zero and the yield pressure the scale distance traversed averaged about 80 divisions, but varied of course with the size of the cylinder. As has been stated above, the complete instrument comprises two lateral extensometers of identical design except in so far as the levers and frames differed in order to allow one to pass through the other.

No attempt was made to determine the absolute magnitude of the extensions. The only, but also the essential, requirement was that the scale readings should be accurately proportional to the extensions.

In carrying out a test, the cylinder was rigidly held with its axis vertical in a vice mounted on a concrete pedestal. Resilient pads were provided to damp out tremors from the ground which otherwise interfered with the reading of the scales. The pressure connections having been made and the extensometers mounted, the whole apparatus was allowed to stand for several hours, generally overnight, in order to equalise the temperature. The tests were carried out in a room in which temperature variations were extremely slight; changes of temperature, however, which are the same in the cylinder and extensometer are without influence, as those parts of the extensometers which by expansion or contraction might affect the readings were made of steel.

The pressure was raised by intervals of 1 ton/square inch until within about 1 ton/square inch of the yield point, readings being taken on both extensometers. Thereafter the intervals were 0·1 ton/square inch, until the yield point was reached. The scale readings of both extensometers were very closely proportional to the pressure during this period. The yield point was noticed at once, and unmistakably, by the fact that either both extensometers would record, for the last increment of pressure, a greatly increased extension or, as was more usually the case, one extensometer would record an increased extension while the other registered a diminution. Some time elapsed before the readings became stationary after the yield point was passed, but after a period of 15 minutes the further movement was negligible in amount. The pressure was then increased by stages of from $\frac{1}{4}$ to $\frac{1}{2}$ ton/square inch and the readings of each extensometer taken after 15 minutes had elapsed. The curves obtained by plotting the individual extensometer readings (as the ratio of the reading to that at the yield point) against the corresponding pressure ratio were irregular, sometimes diverging and sometimes crossing each other. In the case of the thinner cylinders of steel A one curve turned sharply back from the yield point and continued in a retrograde direction. In most cases, however, when the mean of the two readings was taken, a smooth and regular curve was obtained. Examples are shown in fig. 19. The relation between pressure and extension below the yield point being almost perfectly linear, only the upper portion of the elastic line is shown, for reasons of space. It will be seen that the plastic line follows closely the shape of the theoretical curves. The value of μ can in each case be estimated from the position relative to the calculated curves, and the values thus obtained are given in Table VI for all the cylinders tested. Six only of the cylinders of the first group were overstrained, as it was desired to retain the rest for further experiments. The divergence of the extension curves in the case of the two thinnest cylinders of steel A was so great that the mean could not be considered reliable, and they have therefore not been included in the results.

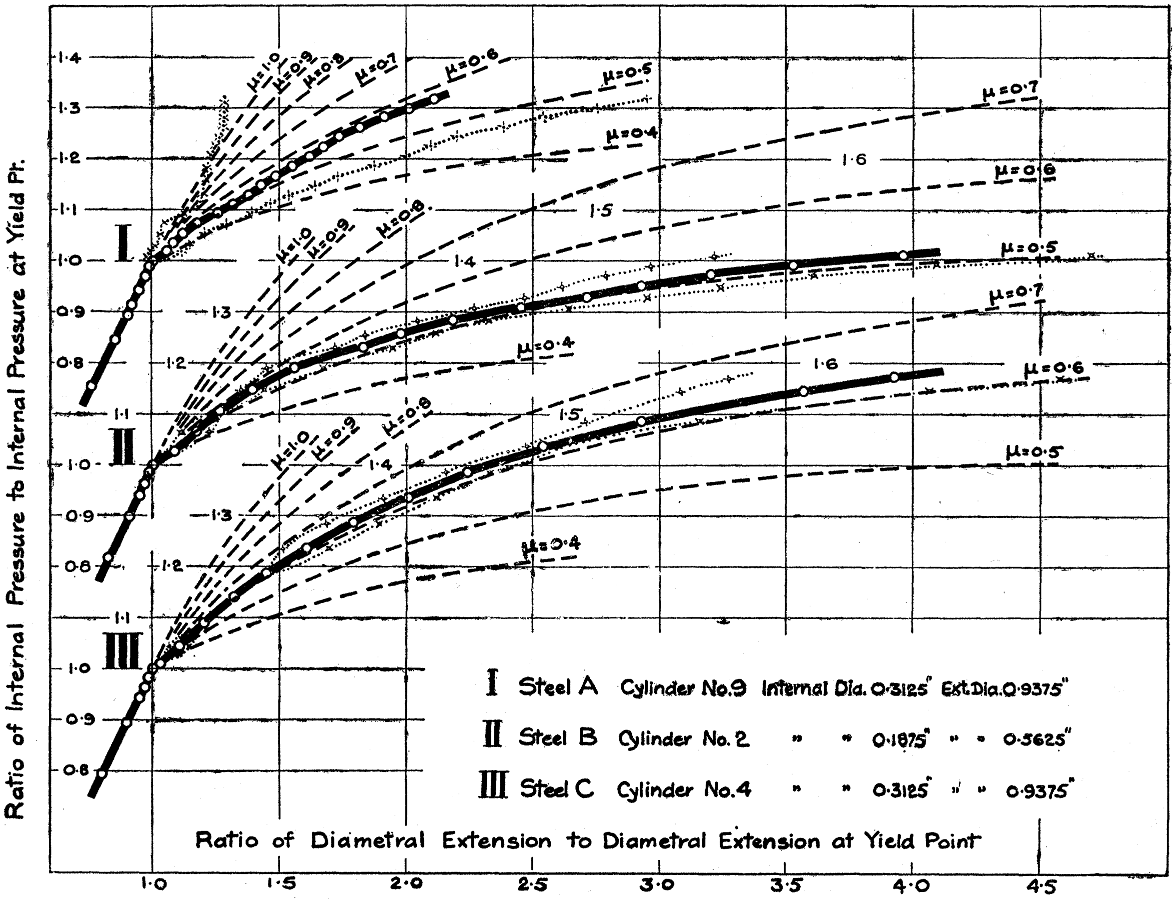

FIG. 19.—Pressure-Extension Diagrams for Internal Pressure.

7. *General Discussion of the Results.*

Conditions at Initial Yield Point.—In Table VII are given, for purposes of comparison, the average values of both the maximum shear stress and the maximum strain energy existing in the material at the initial yield point. In view of the scale effect observed with the smaller holes, only the cylinders having an internal diameter of $\frac{5}{16}$ inch have been included in this table, this being the largest size used in the case of steel A.

The strain energy figures may first be considered. The values for torsion and internal pressure are not greatly different, but those for flexure deviate rather widely from them, even when the square root is used for the comparison. It is noteworthy that there is close agreement between the results in torsion and transverse tension. There is, however, no apparent reason why, in connection with strain energy, torsion should correspond more closely with transverse tension than with longitudinal tension.

In turning now to the shear stress figures it will be seen that the agreement among

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TABLE VII.—Comparison of Values of Shear Stress and Strain Energy per unit volume at Yield Point for different types of Stress Distribution.

(Average values.)

	Maximum Shear Stress. (tons per square inch).			Maximum Strain Energy per Unit Volume. (inch-tons per cubic inch).		
	Steel A.	Steel B.	Steel C.	Steel A.	Steel B.	Steel C.
Tension (longitudinal)	10·4	12·6	10·8	0·0161	0·0237	0·0174
Tension (transverse)	9·75	11·2	9·45	0·0142	0·0187	0·0133
Torsion	12·4	13·9	11·9	0·0147	0·0185	0·0136
Flexure	12·1	14·1	12·1	0·0219	0·0296	0·0219
Internal pressure ($\frac{5}{16}$-in. internal diameter)	12·2	13·0	11·4	0·0150	0·0163	0·0124

the non-uniform distributions is much closer, although the values are greater than in simple tension. In considering failure by shear stress it is necessary to have regard to the position of the planes of maximum shear in each case. In flexure, for instance, it should be the same as those in longitudinal tension, and the comparison should be made accordingly. In the cylinders, on the other hand, the surfaces of maximum shear are parallel to the axis and inclined at 45° to transverse diameters, and the conditions of shear are therefore the same as those of the transverse tension tests. The torsion tests, however, cannot be referred with this degree of certainty to either longitudinal or transverse tension, since the planes of maximum shear in torsion are the planes of zero shear in both tension tests. The most reasonable course to take in this case is to make the comparison with the average of the tension values. Calculating, on this basis, the ratio of the shear stress in the non-uniform distributions to the tension values, we obtain the following results.

TABLE VIII.—Ratio of Shear Stress at failure under Non-uniform Stress to Shear Stress in Uniform Tension.

——	Steel A.	Steel B.	Steel C.
Torsion	1·23	1·17	1·18
Flexure	1·17	1·12	1·12
Internal Pressure	1·25	1·16	1·19

The agreement between the torsion and cylinder tests will be seen to be very close. The flexure tests give only slightly lower values, and the impression is very strong that shear stress is, in fact, the determining factor in failure. It remains, however, to consider why the above ratios are greater than unity, in other words why the shear stress at failure should be greater when the stress distribution is not uniform.

It has already been stated, in the introduction, that the idea of reinforcement by understressed material is not convincing, as it involves an action, *prior to elastic breakdown*, inconsistent with the theory of elasticity. The results rather suggest the possibility that the initial dislocation resulting in elastic breakdown takes place within the body of the material in preference to the surface; that, whilst having the same elastic properties, in other words offering the same resistance to elastic deformation, the surface material may possess a greater resistance to the dislocation by virtue of its being at the surface, as distinct from any mechanical or heat treatment it may receive. It is important to notice that such a property, if it existed, would not be revealed by the ordinary tension test of solid specimens. In that test there is no indication as to whether elastic breakdown begins at the surface or in the interior. Since the stress is uniform, failure occurs when the stress reaches the critical value in *any* part of the material, and owing to the immediate reduction in stress in that part, and the consequent increase in the remainder, complete failure is practically instantaneous over the whole cross-section. If, for example, it is assumed that the surface has an elastic limit 20 per cent. greater than the remainder, and that at the yield point there is a drop of 30 per cent. in stress in the body-material, it will not be until the specimen is so reduced in size that the surface layer occupies three-fifths of the whole cross-section that any effect on the yield point would be produced.

If, however, there is such a surface layer, we should expect it to be revealed by precisely such tests, using non-uniform distributions, as those described in the present paper. Failure would take place when the stress, not at the surface, but at a certain depth in the material, reached the critical value. Its existence would be sufficient to account, not only for the increase in the yield point in the case of torsion and internal pressure, which might otherwise be explained by a particular theory of failure under compound stress, but also for that of flexure. It appears, moreover, to be the only way of accounting for the scale effect observed in the cylinders. It may reasonably be supposed that the surface effect will extend inwards by an amount which depends on the microstructure only, and which will be approximately constant for material of the same composition and heat treatment. It will therefore not conform to the conditions of similarity, but will have a relatively greater effect in the cylinders of smaller internal diameter.

It will be instructive to calculate the thickness which would be required for such a surface layer in order to give the results obtained in the present investigation. For this purpose it will be assumed that failure actually takes place in the non-uniform distributions at values of the shear stress given by the tension tests having regard, as before, to the particular planes of maximum shear, and the problem is to determine at what depth below the surface such stresses are to be found when the yield point was observed.

In view of the linear distribution of stress in torsion and flexure, the calculation in these cases is one of simple proportion. For the cylinders it is easily seen that the shear stress at radius a due to internal pressure P is given by

$$s = \mathrm{P} : \frac{a_1^2}{a_1^2 - a_0^2} \cdot \frac{a_0^2}{a^2},$$

where a_0, a_1 are the internal and external radii, so that

$$\frac{a^2}{a_0^2} = \frac{\mathrm{P}}{s} \cdot \frac{a_1^2}{a_1^2 - a_0^2},$$

from which the radius a, and therefore the thickness $a - a_0$, is readily obtained, s being the shear stress at failure in transverse tension. The following results are obtained in this manner for steel A (for which the greatest number of tests are available).

TABLE IX.—Estimated depth of Hypothetical Surface Layer.

	Extreme Values.	Average Value.
	inches.	inches.
From torsion tests	0·0208	0·0188
	0·0169	—
From flexure tests	0·0160	0·0144
	0·0127	—
From cylinder tests—		
Internal diameter, 0·312 inch	0·0202	0·0188
	0·0173	—
" " 0·222 inch	0·0178	0·0169
	0·0158	—
" " 0·125 inch	0·0138	0·0135
	0·0131	—

The figures are of the same order, and the differences are not great when it is considered that a variation in the value of the shear stress at the yield point of 5 per cent. will alter the figures by amounts from 20 to 30 per cent.

The results of the yield point determinations are therefore consistent with a supposition that a surface layer exists, having an elastic limit greater than that in the body of the metal. It will account fully for the apparently increased yield points in all the cases of non-uniform stress examined, and for the scale effect observed in the cylinder tests. The hypothesis is, however, by no means free from difficulty. A consequence, for example, should be a scale effect also in torsion and flexure tests. This is a matter for further investigation, all the specimens in the present series being of the same size. The author is not aware of any test results which show such an effect, except in the fracture of cast iron by flexure* and this is probably due to a different cause.

* PEARCE, 'J. Iron & Steel Inst.,' vol. 118, p. 73 (1928).

It is noteworthy, however, that a scale effect has been observed by STANTON and BATSON* in impact tests. Although similarity was maintained both in the specimen and in the testing appliances, the energy per unit volume (which has the dimensions of stress) was found to increase as the size was reduced. It appeared, moreover, that the effect was greater in steel of lower carbon content, a tendency also shown by the present tests. No explanation of the effect was given, but it was suggested that it might possibly lie in the fact that similarity had not been extended to the microstructure.

Conditions during Overstrain.—The investigation has shown that in the initial stages of overstrain of mild steel in torsion, flexure, and by internal pressure, there is a considerable reduction in the stress carried by the overstrained portions, corresponding to that observed in the tensile stress-strain diagram, in the region immediately following the yield point. The values of the shear stress accompanying the plastic strain, estimated from the load-displacement diagrams in each case by the method described in the paper, are summarised in Table X.

TABLE X.—Comparison of Values of Shear Stress in Overstrained Regions for different types of Stress Distribution.

(Average values.)

——	Steel A.	Steel B.	Steel C.
	tons/sq. in.	tons/sq. in.	tons/sq. in.
Tension (longitudinal)	7·5	8·5	7·7
Tension (transverse)	7·6	7·5	7·4
Torsion	8·0	8·5	7·8
Flexure	7·4	8·3	7·8
Internal pressure	6·8	7·2	7·0

It will be seen that a reasonable degree of uniformity exists in the figures obtained for each steel, when the limitations of the theoretical treatment, and the simplicity of the assumptions regarding the yielding process upon which it is based, are remembered. The agreement between the shear stresses in longitudinal tension and flexure is particularly close. In these two cases the stresses are of the same character, differing only in distribution, and the planes of maximum shear are also the same. Agreement should therefore be expected to be close if the method of analysis is adequate. The figures for torsion also agree well with those for longitudinal tension in steels B and C, but in steel A the value for torsion is rather greater. It will be noticed, however, that the shear stress in transverse tension is less for B and C than in longitudinal tension, while for A it is slightly greater. It has been previously stated that the planes of maximum shear in torsion are those of zero shear in both the tension tests, and therefore the

* 'Proc. Inst. C.E.,' vol. 211, p. 67 (1920).

correspondence is less complete. The values for internal pressure are definitely, but not greatly, less than those in transverse tension, to which they correspond.

In considering the relation between these figures it is necessary to recall that in the non-uniform distributions they were obtained on the assumption that each successive layer yielded at the same value of the shear stress; that is, at the shear stress which caused the initial breakdown of the surface layer. The hypothesis of a surface layer of higher elastic limit will not affect the values given above for torsion and flexure since, as has been previously stated, they are calculated from the conditions under the larger displacements when the portion remaining elastic contributes little to the torque or moment of resistance. This is, however, not true in the case of internal pressure. A higher value, and one more nearly approaching the transverse tension value, would be obtained by the supposition that each layer in turn yielded at a stress less than that in the surface layer when the process began. But it must be remembered that the uncertainty regarding the distribution of axial stress in this case may be sufficient to account fully for the differences observed. The general consistency of the results indicates that the stress distribution during overstrain is represented fairly accurately by the diagrams shown in figs. 7 and 13.

Although the main purpose of this paper has been to record the results of experiments, it is desirable to form, if possible, some picture of the action taking place within a crystal grain which would be consistent with the externally observed phenomenon of stress reduction. It is well known that the plastic deformation of metals is accompanied by slip over certain planes within the crystal grains. The nature and mechanism of the slip is imperfectly understood, but the fact of its existence has received abundant confirmation since the pioneer work of EWING and ROSENHAIN was carried out. It was formerly thought that the crystal structure within the grain offered a high resistance to slip, and that the elastic strength as externally observed could be attributed to this resistance. The knowledge which has since become available regarding the properties of large single crystals has necessitated a revision of this view. It is now known that the resistance which the undisturbed crystal lattice itself offers to slip, at all events in a large crystal, is very small. According to the experiments of EDWARDS and PFEIL* the elastic limit of a single crystal of iron is reached at a stress below $2\frac{1}{2}$ tons per square inch. It has become necessary therefore to regard the crystal boundary as the main source of elastic strength in an aggregate. The structure at the surface of separation of two adjacent crystal grains is a matter of controversy. According to one theory,† the atomic arrangement in the boundary material is of an irregular character in which slip, as understood in a regular lattice, is impossible, with the result that it possesses both hardness and brittleness. It appears, moreover, to have the power of inhibiting slip in the regular arrangement relatively remote from the boundary.

Whatever may be the true nature of the boundary layer it will conform to the general

* 'Journal, Iron and Steel Inst.,' vol. 112, No. 2 (1925), p. 79.

† ROSENHAIN, 'International Congress for Testing Materials,' 1927, vol. 1, p. 59.

elastic deformation which takes place below the yield point, a deformation which is reversible, and continuous both in space and time. The volume occupied by the boundary material must be relatively very small, and consequently the deformation of the lattice in each crystal will be sensibly the same as that applied to the aggregate as a whole. Now it was found by EDWARDS and PFEIL that the modulus of elasticity of the single crystal of iron differed little from that ordinarily observed in iron and steel aggregates, and it seems probable therefore that the lattice of each crystal will carry a stress approximately the same as that deduced from the general strain. There is no means of estimating the stress residing in the boundary layers. The modulus for the irregular structure may itself be variable, but the small relative volume concerned makes it possible for the stress to vary between wide limits without affecting appreciably the magnitude of that carried by the regular lattices.

It may reasonably be inferred, therefore, that the breakdown at the yield point is initiated in the boundary region, and that the slip which takes place in the regular lattice is a consequence of the dislocation.* It is not necessary, for the purpose of this discussion, to consider the nature of the dislocation; but there is little doubt that the displacements arising from it may be considerable, and the lattice will readjust itself to the new configuration by the process of slip. But it is essentially a discontinuous process both in time and space, and is irreversible. It is reasonable to expect that the reversible elastic deformation which existed both in the boundary material and in the crystal body will not be retained without change after the dislocation has taken place, even in parts which may be relatively remote from the site of the dislocation. The view here put forward is that a uniformly increasing strain given to the metal as a whole will be communicated to each element—which may be regarded as comprising at least one crystal grain—without appreciable change. Before any dislocation takes place in the element the displacements will be entirely reversible. If we assume that this element is on the point of breakdown, the reversible distortion will have reached its maximum value. A dislocation, and consequent slip, now occurs. As there will, at that instant, be no change in the strain of the element as a whole, the irreversible displacements, since they are discontinuous, must necessitate a reduction in the displacements which are reversible, and upon which the stress in the element depends. The consequence is a relief of stress, and the amount of the relief will depend upon the magnitude of the displacement associated with the dislocation in relation to the pre-existing elastic deformation.

The view of MUIR and BINNIE† regarding the discontinuous nature of the yielding process is that each portion of the material is either in the condition represented by A (fig. 2) or by that of B, no intermediate state being possible. A similar view has been

* The suggestion that the formation of slip bands within the crystal grains may be a secondary rather than the primary process associated with the phenomenon of plastic yield was made to the author by Professor B. P. HAIGH.

† *Loc. cit.*

expressed by NAKANISHI* who, however, regards the states C and D as alone possible for equilibrium over the range of strain represented by CD. In each, the actual strain in the whole mass at any instant is regarded as dependent upon the relative proportions of material in the two states, but in neither case are these states referred to the conditions which may have arisen in the crystal. The magnitude of the displacement in an individual slip—meaning now by the term "slip" the dislocation either in the regular lattice or in the boundary region—may differ very greatly for different metals and different crystal structures. The above suppositions will correspond to a comparatively large displacement, and this may well be the case with metals such as mild steel, showing a pronounced yield point. The local relief of elastic strain consequent upon the slip may therefore be substantial.

It has already been suggested that in the irregular boundary material the stresses may vary over a very wide range from one point to another at any given instant. It may be expected therefore that the breakdown will occur first where the highest stresses are found, and where the configuration is most favourable to the displacements required. The reduction of stress which follows will throw additional load upon neighbouring grains, and the process of yield will be propagated with great rapidity. At any given moment there will be a considerable variation in stress in the regular lattice from grain to grain. It will be small in a crystal which has just slipped. It will be large in one which is on the point of slipping. Intermediate values will be found in crystals in which slip has taken place at some previous instant, and in which the continued strain is renewing the process of elastic deformation, which will gradually increase until fresh dislocations occur. The whole process is consistent with a reduced external load, or *average* stress, and will probably continue until the dislocations and slip planes are sufficient in number to restrict to some extent the freedom of the intervening atomic structure, when the yielding process will end, and a rising stress will accompany further strain.

The fact that after removal and re-application of load during the yielding process flow again takes place at the lower yield stress (fig. 6) may readily be explained on this view. Removal of load will be accompanied by removal of the elastic deformation, or even reversal in the case of grains carrying little stress at the moment of removal. The displacements due to slip will be unaffected except possibly for small adjustments that may take place. Re-application of load will simply restore the pre-existing elastic strain, and crystals which were then on the point of failing will again be in the same state.

As to the magnitude of the stress reduction which, according to the view here put forward, may accompany the dislocation within an individual grain, nothing definite can be stated. There are some grounds for supposing that the stress may even be reduced to zero. The fact, for instance, that when the external load is reversed plastic flow may, as shown by ROBERTSON, commence at zero load, would suggest that this

* 'Journal of the Society of Mechanical Engineers of Japan,' vol. 31, No. 130 (1928).

is the case. The simple removal of load would be sufficient to bring the elastic deformation of such grains to the critical value in the reverse direction.

Whether or not the above view of the process of yielding in mild steel is the correct one, there is no doubt that the initial stage of plastic strain, both in uniform or non-uniform distributions of stress, takes place under an average stress very much less than the initial yield stress. It is not improbable that this property of relieving itself, so to speak, of stress when slightly overstrained, and thus levelling out concentrations of stress, lies at the root of the well-known reliability of this material for all structural purposes.

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